A STUDY ON POISSON PROCESS IN STOCHASTIC PROCESS

N. Monika[1] and B. AmrinTaj[2]
Asst. Professor[1], M.Phil., scholar
Department of Mathematics
Adhiyaman Arts and Science College For Women, Uthangarai, TamilNadu

Abstract: In this paper we study about the Poisson process in various time such as continuous type and also in discrete time with an inter arrival time in various space state.

Keywords: Poisson process, continuous time space, discrete time space, inter arrival time, waiting time distribution.

INTRODUCTION

We shall deal with some stochastic processes in continuous time and with discrete state space which a part from their theoretical importance, play an important role in the study of a large number of phenomena. One such process in Poisson process to which we shall confine ourselves here. To fix our ideas consider a random event to such as

(i) Incoming telephone calls (at a switch board)
(ii) Arrival of customers for service (at a counter)
(iii) Occurrence of accidents (at a certain place)

Let us consider the total number N(t) of occurrences of the events E in an interval of duration t. (I,e), (if we start from an initial epoch (or instant) t = 0, N(t) will denote the number of occurrences up to the epoch or instant) t (more precisely to t+0)

For example, if an event actually occurs at instants of time t_1 , t_2, t_3,….. then N(t) jumps abruptly from 0 to 1 at t = t_1, from 1 to 2 at t = t_2 and so on. The values of N(t) are observed values of the random variable N(t). Let $P_n(t)$ be the probability that the random variable N(t) assume the value n,

$$P_n(t) = \Pr\{N(t) = n\}$$

This probability is a function of the time t, since the only Poisson values of n are n = 0,1,2,…..

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

Thus $\{P_n(t)\}$ represents the probability distribution of the random variable N(t) for every value of t. the family of random variables {N(t), t≥0} is a stochastic process, which servers as a mathematical model for a wide range of empirical phenomena with remarkable accuracy. The justification for this is based on the concept of rare events. We proceed to show that, under certain conditions N(t) follows Poisson process with mean λ t (λ being a constant). In case of many empirical phenomena, these conditions are approximately true and the corresponding stochastic process {N(t), t≥0} follows the Poisson law. A stochastic process {X(t)} with integral- valued state space [associated with counting (one by one) of an events such that as t increases]. The cumulative count can only increase is called a counting or point process. Cox and Isham (1980) deal this subject exhaustively Poisson process is such a process.

POISSON PROCESS

A stochastic process {N(t).t≥0} is said to be a counting process. If N(t) represents the total number of events that have occurred up to time t. hence, a counting process N(t) must satisfy

(i) N(t)≥0
(ii)N(t) is integer valued
(iii) If s<t, then N(s)≤N(t)
(iv) For s<t, N(t) – N(s) equals the number of events that have occurred in the interval (s,t].

A counting process is said to possess independent increments if the numbers of events that occur in disjoint time intervals are independent. For example, this means that the number of events that have occurred by time t (that is N(t)) must be independent of the number of events occurring between t and t + s (that is, N((t + s) – N(t)). A counting process is said to possess stationary increments. If the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the numbers of events in the interval (t_1+s, t_2+s] (that is N(t_2+s)-N(t_1+s)) has the same distribution as the number of events in the intervals (t_1,t_2] (that is , N(t_2)-N(t_1)) for all t_1<t_2, and s>0. One of the most important types of counting processes is the Poisson process, which is defined as follows.

POISSON PROCESS (1)

The counting process {N(t),t≥0} is said to be Poisson process having rate λ , λ >0, if

(i) N(0) = 0.
(ii)The process has independent increments.
(iii) The number of events in any interval of length t is Poisson distributed with mean λ t, that is, for all s,t≥0.

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n = 0,1,2,...$$

Order of h

The function f is said to be o(h) if $\lim_{h\to 0}\frac{f(h)}{h}=0$. We are now in a position to give an alternative definition of a Poisson process

POISSON PROCESS (2)

The counting process {N(t),t≥0} is said to be Poisson process with rate λ, λ >0, if

(i) N(0) = 0

(ii)The process has the stationary and independent increments

(iii) P(N(h)=1)= λ h + o(h)

(iv) P{N(h)≥2}=o(h)

THEOREM 1

The counting process {N(t),t≥0} is said to be a Poisson process having rate λ, λ >0, if

(i) N(0) = 0

(ii)The process has independent increments

(iii) The number of events in any interval of length t is Poisson distributed with λ t. that is, for all s, t≥0

$$P\{N(t+s)-N(s)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}, n=0,1,2,....$$

And the counting process {N(t),t≥0} is said to be a Poisson process with rate λ, λ >0, if

(i) N(0) = 0

(ii)The process has stationary and independent increments

(iii) P(N(h)=1)= λ h + o(h)

(iv) P{N(h)≥2}=o(h) are equivalent

Proof

We first show that Poisson process (1) implies Poisson process (2).

Let $P_n(t)$ = P{N(t)=n}

We derive a differential equation for $P_0(t)$ in the following manner

$P_0(t + h)$ = P{N(t + h)=0}

= P{N(t)=0,N(t + h)-N(t)=0}

= P{N(t)=0}p{N(t + h)-N(t)=0}

= $P_0(t)[1-\lambda h + o(h)]$

Where the final two equations follow from assumption (ii) and the fact that (iii) and (iv) imply that P{N(h)=0}=1-λ h+o(h).

Hence,

$$\frac{P_0(t+h)-P_0(t)}{h}=-\lambda P_0(t)+\frac{0(h)}{h}$$

Letting h→0 yields

$$P_0'(t)=-\lambda P_0(t) \quad \text{(or)} \quad \frac{P_0'(t)}{P_0(t)}=-\lambda$$

Which implies, by integration,

Log $P_0(t)$ = - λ t+c (or)

$P_0(t) = k\, e^{-\lambda t}$

Since $P_0(0)$ = P{N(0)=0} =1

We arrive at

$$P_0(t) = e^{-\lambda t} \qquad (1)$$

Similarly,

For n≥1.

$P_n(t+h)$ = P{N(t+h) = n}

$= P\{N(t)=n, N(t+h)-N(t)=0\}$

+ P{N(t)=n-1, N(t+h)-N(t)=1}+P{N(t+h)=n.N(t+h)-N(t)>2}

However, by (iv), the last term in the above is o(h) hence

We obtain

$$P_n(t+h)=P_n(t)P_0(h)+P_{n-1}(t)P_1(h)+o(h)$$

$= (1-\lambda h)P_n(t)+\lambda hP_{n-1}(t)+o(h)$

Thus,

$$\frac{P_n(t+h)-P_n(t)}{h}=-\lambda P_n(t)+\lambda P_{n-1}(t)-\frac{o(h)}{h}$$

Letting h→0,

$P_n(t) = -\lambda P_n(t)+\lambda P_{n-1}(t)$ or equivalently,

$$e^{-\lambda t}[P_n'(t)+\lambda P_n(t)]=\lambda e^{\lambda t}P_{n-1}(t)$$

Hence,

$$\frac{d}{dt}(e^{\lambda t}P_n(t))=\lambda e^{\lambda t}P_{n-1}(t) \qquad (2)$$

Using equation (1)

$$P_0(t) = e^{-\lambda t}$$

We have n=1

$$\frac{d}{dt}(e^{\lambda t}P_n(t))=\lambda e^{\lambda t}P_{n-1}(t)$$

$$\frac{d}{dt}(e^{\lambda t}P_1(t))=\lambda$$

(or) $P_1(t)=(\lambda t+c)e^{-\lambda t}$

Since, $P_1(0)=0$,

Yields $P_1(t)=\lambda t e^{-\lambda t}$

Show that $P_n(t)= e^{-\lambda t}\frac{(\lambda t)^n}{n!}$

We use mathematical induction and hence first assume it for n-1. Then by equation (2)

$$\frac{d}{dt}(e^{\lambda t}P_n(t))=\lambda e^{\lambda t}e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$\frac{d}{dt}(e^{\lambda t}P_n(t))=\lambda\frac{(\lambda t)^{n-1}}{(n-1)!}$$

Implies that

$$e^{\lambda t}P_n(t)=\frac{(\lambda t)^n}{n!}+c$$

Since $P_n(0)=P\{N(0)=n\}=0$

$$P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

INTERARRIVAL AND WAITING TIME DISTRIBUTIONS

Consider a Poisson process and let X_i denote the time of the first event. Further, for n≥1, let X_n denoted the time between the (n-1)th and the nth event. The sequence $\{X_n, n\geq 1\}$ is called the sequence of inter arrival times. We shall now determine the distribution of the X_n. to do so we first note that the event $\{X_1>t\}$ takes place if and only if no events of the Poisson process occur in the interval [0,t] and thus

$$P\{X_1>t\}=P\{N(t)=0\}=e^{-\lambda t}$$

Hence, X_1 has an exponential distribution with mean $1/\lambda$. To obtain the distribution of X_2 condition on X_1. This gives

$$P\{X_2>t|X_1=s\} = P\{0 \text{ events in } (s,s+t]|X_1=s\}$$
$$= P\{0 \text{ events in } (s,s+t]\} \quad \text{(by independent increments)}$$
$$= e^{-\lambda t} \quad \text{(by stationary increments)}$$

Therefore, from the above we conclude that X_2 is also an exponential random variable with mean $1/\lambda$, and furthermore, that X_2 is independent of X_1, Repeating the same argument yields the following

RESULT

Let X_n, n = 1,2,….. are independent identically distributed exponential random variables having mean $1/\lambda$.