NORMALITY OF MEROMORPHIC FUNCTIONS PERTAINING SHARED VALUES

***Abstract:** Here n, k is certain whole numbers considering n ≥ 2k + 4, a ≠ 0 is complex number, and F is gathering of functions meromorphic in D space, for every f ∈F, fn + af(k) and gn + ag(k) share b, and zeros have extend minimum of k + 1, at that point F is ordinary in D*

***Keywords:** meromorphic function; normal family; shared value.*

1. Introduction and results

In this exploration, C is indicated as entire complex plane. Here f is meromorphic worark in area D ⊂C. For $a \in \mathbb{C}$, set $\overline{E}_f(a) = \{z \in D : f(z) = a\}$. The meromorphic capacities g and f share esteem a with the end goal that $\overline{E}_f(a) = \overline{E}_g(a)$ in D. At the point when a = ∞, zeros of f − a methods shafts of *f*.

Let F a gathering of meromorphic functions characterized on D ⊂C. F is said to be typical on D, in the feeling of Montel, if for any grouping f_j ∈F there exists a subsequence fnj merges circularly locally consistently on D, to a meromorphic work or ∞ (see [1], [2], [3]).

In Bloch's standard, all condition which reduces a meromorphic work in plane C to a consistent, changes a gathering of meromorphic works in an area D to ordinary. It is vital to discover ordinariness criteria in shared qualities. Schwick[4] demonstrated that a gathering of meromorphic works in area is typical if in which all capacity shares 3 unmistakable limited complex numbers with its first subordinates. Sun[5] exhibited a gathering of meromorphic works in space is typical if in which each match of capacities share 3 settled discrete qualities, which was ad libbed variant of Montel's Normal Criterion [6] by shared qualities. Numerous outcomes on ordinariness criteria with respect to shared qualities was discovered [7-9].

In 2008, Zhang[10] proved

Theorem A. (see [10]). Let F be a gathering of capacities meromorphic in a space D, n be a positive whole number and a, b be two constants with the end goal that a =6 0,∞ and b 6= ∞. In the event that n ≥ 4 and for every f and g in F, $f' - af^n$ and $g' - ag^n$ share the esteem b, at that point F is ordinary in D.

In this paper,we replace *f'* by $f^{(k)}$ in Theorem A and obtain the following theorem.

Theorem 1. Let n, k be a positive whole numbers fulfilling n ≥ 2k + 4, a 6= 0,∞ and b =6 ∞ be intricate numbers, and let F be a gathering of capacities meromorphic in an area D. In the event that for each *f,g* ∈F, $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share b, and every one of the zeros have assortment in any event k + 1, at that point F is ordinary in D.

Example: Let $D = \{z : |z| < 1\}$ and F = $\{f_n\}$ where

$$f_n(z) = \frac{1}{n\sqrt[4]{z}}, z \in D, n = 1, 2, 3, \ldots$$

Clearly $f_n'' + f_n^9 = \frac{5n^8+5}{8n^9\sqrt[4]{z^9}}$. So for each pair m, n, $f_n'' + f_n^7$ and $f_m'' + f_m^7$ share 0 in *D*, but F is not normal at the point *z* = 0 since $f_n^\sharp(\frac{1}{n^4}) = \frac{n^4}{4(1+n)} \to \infty (n \to \infty)$. This example show that Theorem 1 is not valid if *f* doesn't satisfy that all zeros have multiplicity at least *k* + 1.

2.Lemmas

In this section, some lemmas were illustrated.

Lemma 2.1([8]). F is a group of functions meromorphic on unit disc, all of zeros have multiplicity with minimum of *k*, and there exists $A \ge 1$ so that

$|f^{(k)}(z)| \le A$ whenever $f(z) = 0$. Then if F is not normal, there exist, for each $0 \le \alpha \le k$,

a) a number $0 < r < 1$;
b) points z_n, $|z_n| < r$;
c) functions $f_n \in F$; and
d) positive numbers $\rho_n \to 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi$ $_2$) locally uniformly with respect to the spherical metric, where *g* is a nonconstant meromorphic function on C, all of zeros have multiplicity with minimum of *k*, such that $g^\#(\xi) \le g^\#(0) = kA + 1$. Particularly, *g* has instructed at most two; and, in case *g* is entire function, as exponential type. **Lemma 2.2** [11] Let *f* be a meromorphic function, then

$$T(r,f) < \overline{N}(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f'-1}) - N(r,\frac{1}{f''}) + S(r,f), \qquad (2.1)$$

where

$$S(r,f) = 2m(r,\frac{f'}{f}) + m(r,\frac{f''}{f'-1}) + \log\left|\frac{192f(0)(f(0)-1)}{f'(0)}\right|.$$

Lemma 2.3 Let *f* be a transcendental meromorphic function, *n* and *k* be two integers and $n \ge 2k + 4$, and all zeros of *f* are of multiplicity greater than *k* + 1, then $f^n + af^{(k)}$ assumes every finite complex value *b* infinitely often.

Proof: Let $\psi(z) = \frac{af^{(k)}(z)-b}{f^n(z)}$, and suppose that $\psi(z) = -1$ has only finite number of roots. Then by Lemma 2.2

$$T(r,\psi) < \overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) + \overline{N}(r,\frac{1}{\psi+1}) + S(r,\psi). \tag{2.2}$$

Now poles of $\psi(z)$ occur only at zeros of $f(z)$, and those poles which are not instantaneously zeros of $af^{(k)} - b$ have multiplicity minimum of n. Zeros of $\psi(z)$ can occur only at zeros of $af^{(k)} - b$ which are not poles of $f(z)$. Thus

$$\begin{aligned}\overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\le \frac{1}{n}N(r,\psi) + \overline{N}(r,\frac{1}{af^{(k)}-b}) + \overline{N}(r,f)\\ &\le \frac{1}{n}T(r,\psi) + T(r,f^{(k)}) + \overline{N}(r,f). \end{aligned}\tag{2.3}$$

By the first fundamental theorem,

$$\begin{aligned}\overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\le \frac{1}{n}T(r,\psi) + T(r,f) + (k+1)\overline{N}(r,f) + S(r,f)\\ &\le \frac{1}{n}T(r,\psi) + (k+2)T(r,f) + S(r,f).\end{aligned}\tag{2.4}$$

Take (2.4) into (2.1),

$$(1-\frac{1}{n})T(r,\psi) < (k+2)T(r,f) + S(r,f). \tag{2.5}$$

On the other hand,

$$\begin{aligned} nT(r,f) &= T(r,f^n) = T(r,\frac{af^{(k)}-b}{\psi})\\ &\le T(r,f^{(k)}) + T(r,\psi) + O(1)\\ &\le (k+1)T(r,f) + T(r,\psi) + S(r,f), \end{aligned}$$

that is

$$(n-k-1)T(r,f) \le T(r,\psi) + S(r,f). \tag{2.6}$$

Combine (3.5) and (3.6) obtaining

$$(1-\frac{1}{n})T(r,\psi) < [\frac{k+2}{n-k-1} + O(1)]T(r,\psi) \tag{2.7}$$

which contradicts with the condition $n \ge 2k+4$.

Thus proved Lemma 2.3.

Lemma 2.4 f *is* nonconstant rational function, n and k be two integers and $n \ge 2k+4$, and all zeros of f are of multiplicity higher than $k+1$, then $f^n + af^{(k)}$ has minimum of 2 distinct zeros.

Proof: Case 1. If $f^n + af^{(k)}$ has no zeros, it is easy to see that f is not a polynomial, then f is rational function but not a polynomial.

Let $f = \frac{P}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_t)^{m_t}} = \frac{P}{Q}$, $f^{(k)} = \frac{P_1}{Q_1}$. We denote $p = degP$, and $q = degQ$, then $degQ_1 = q + kt$, $degP_1 = p + k(t-1)$.

$$f^n + af^{(k)} = \frac{P^nQ_1 + aP_1Q^n}{Q^nQ_1}. \tag{2.8}$$

$deg(P^nQ_1) = np + q + kt$, $deg(P_1Q^n) = p + k(t-1) + nq$.

If $p \ge q$, then $np + q + kt - (p + k(t-1) + nq) = (n-1)(p-q) + k > 0$, that is $deg(P^nQ_1) > deg(P_1Q^n)$; If $p < q$, since $n \ge 2k+4$, then $np+q+kt-(p+k(t-1)+nq) = (n-1)(p-q)+k < 0$, that is $deg(P^nQ_1) < deg(P_1Q^n)$, thus $f^n + af^{(k)}$ have zeros, which is a contradiction.

Case 2. $f^n + af^{(k)}$ has only one distinct zero z_0.

If f is a polynomial, then $f^n + af^{(k)} = A(z-z_0)^l$. From the condition that all zeros of f are of multiplicity greater than $k+1$, it can be deduced that z_0 is the only zero of f, so $f = b(z-z_0)^m, m \ge k+1$, where b is a constant and m is a positive integer.

$$f^{(k)} = bm(m-1)\cdots(m-k+1)(z-z_0)^{m-k} \tag{2.9}$$

and $f^n + af^{(k)} = A^n(z-z_0)^{nl} + c(z-z_0)^{m-k} = (z-z_0)^{m-k}[A(z-z_0)^{nl-m+k} + C]$, (2.10) thus $f^n + af^{(k)}$ has two distinct zeros, contradiction.

So f is a non-polynomial rational function, then assuming

$$f(z) = \frac{A(z-z_0)^m}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \tag{2.11}$$

where A is a constant, and s is a positive integer. By integration (2.4)

$$f^{(k)} = \frac{(z-z_0)^{m-k}g(z)}{(z-z_1)^{n_1+1}(z-z_2)^{n_2+1}\cdots(z-z_s)^{n_s+1}}, \tag{2.12}$$

From (2.11) and (2.12)

$$\begin{aligned} f^n + af^{(k)} &= \frac{A^n(z-z_0)^{nm}}{(z-z_1)^{nn_1}\cdots(z-z_s)^{nn_s}} + \frac{a(z-z_0)^{m-k}g(z)}{(z-z_1)^{n_1+1}(z-z_2)^{n_2+1}\cdots(z-z_s)^{n_s+1}}\\ &= \frac{(z-z_0)^{m-k}[A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}]}{(z-z_1)^{nn_1}(z-z_2)^{nn_2}\cdots(z-z_s)^{nn_s}}. \end{aligned}\tag{2.13}$$

On the other hand, since $f^n + af^{(k)}$ has only one zero,

$$f^n + af^{(k)} = \frac{C(z-z_0)^l}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \quad (2.14)$$

Combining (2.13) and (2.14)

$$C(z - z_0)^l = (z - z_0)^{m-k} g_1(z),$$

where $g_1(z) = A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}$. If $l > m-k$, then $g_1(z)$ has a zero z_0, which is impossible. If $l = m-k$, $g_1(z) = C$, that is $A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k} = C$, thus $nm-m+k = (n-1)N$, where $N = n_1 + n_2 + \cdots + n_s$, thus $(n-1)(N-m) = k$, which is impossible since $n \geq 2k+4$.

The proof of Lemma 2.4 is completed.

3. The Proof of Theorem 1

Assuming that $D = \Delta$, the unit disc. Suppose that F is not normal on Δ. Then by Lemma1, we can find $f_j \in F, z_j \in \Delta$, and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_n^{\frac{k}{n-1}} f_j(z_j + \rho_j\xi)$ converges locally uniformly with respect to the sphericity metric to a nonconstant meromorphic function g on C, all of whose zeros have multiplicity at least k, which satisfies $g^\#(\xi) \leq g^\#(0) = kA + 1$, in particular, g has order at most two.

On every compact subset of C we have

$$\rho_j^{\frac{nk}{n-1}}[f_j^n + af_j^{(k)} - b] = g_j^n(\xi) + ag_j^{(k)}(\xi) - \rho_j^{\frac{nk}{n-1}} b \to g^n(\xi) + ag^{(k)}(\xi). \quad (3.1)$$

If $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$, then g has no poles and g is not a polynomial, thus g is a transcendental entire function. From $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$ we obtain $g^{n-1} = -a\frac{g^{(k)}}{g}$, by the first fundamental theorem,

$$(n-1)T(r,g) = (n-1)m(r,g) = m(r,g^{n-1}) = m(r,-a\frac{g^{(k)}}{g}) = S(r,g),$$

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since $n \geq 2k+4$, we obtain $T(r,g) = S(r,g)$, which is a contradiction, thus $g^n(\xi) + ag^{(k)}(\xi)$ 6≡ 0.

By Lemma 2.3 and Lemma 2.4 we obtain that $g^n + ag^{(k)}$ has minimum 2 distinct zeros.

Next we prove that $g^n + ag^{(k)}$ has only one distinct zero.

Let ξ_0 and ξ_0^* be two distinct zeros of $g^n + ag^{(k)}$. We choose a small $\delta > 0$ such that $D_1 \cap D_2 = \emptyset$, where $D_1 = \xi \in \mathbb{C} : |\xi - \xi_0| < \delta$ and $D_2 = \xi \in \mathbb{C} : |\xi - \xi_0^*| < \delta$.

From (3.1), Hurwitz's theorem implied that there exist points $\xi_j \in D_1$ and $\xi_j* \in D_2$ such that for sufficiently large j

fjn(zj + ρjξj) + afj(k)(zj + ρjξj) = b, fjn(zj + ρjξj∗) + afj(k)(zj + ρjξj∗) = b.

By the assumption of Theorem 1, we see that for each $f_m \in F$

fmn(zj + ρjξj) + afm(k)(zj + ρjξj) = b, fmn(zj + ρjξj∗) + afm(k)(zj + ρjξj∗) = b.

Fix m and let $j \to \infty$, we have $z_j + \rho_j\xi_j \to z_0$, and $z_j + \rho_j\xi_j^* \to z_0$, and

$$f_m^n(z_0) + af_m^{(k)}(z_0) = b.$$

Since the zeros of $f_m^n + af_m^{(k)} - b$ have no accumulation points, we deduce that $z_j + \rho_j\xi_j = z_0$ and $z_j + \rho_j\xi_j^* = z_0$, for sufficiently large j. Hence $\xi_j = \xi_j^* = (z_0 - z_j)/\rho_j$, which contradicts the fact that $\xi_j \in D_1$, $\xi_j* \in D_2$ and $D_1 \cap D_2 = \emptyset$.

Thus we complete the proof of Theorem 1.

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