Maximal and Minimal Beta open set in Topological Space

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Abstract. Minimal open sets for a topology are defined and investigated. They are found to form an Alexandroff space on X. Decompositions of open sets and continuity are provided using minimal open sets. Also minimal regularity and minimal normality are defined and studied. While Hausdorffness implies minimal regularity, the product of normal spaces are found to be minimal normal. we introduce new classes of sets called maximal β-open sets and minimal β -open sets and investigate some of their fundamental properties

Key word and phrases: θ-open, maximal open sets, minimal open sets, minimal closed, maximal θ-open sets and minimal θ-closed.

I.INTRODUCTION.

Now a days topological approaches are being investigated in a big way in various diverse field such as computer graphics, evolutionary theory, robotics etc.[6, 9, 16] to name a few. One such approach to computer graphics utilizes finite, connected order topological space[8]. In a finite topological space, the intersection of all open neighbourhoods of a point p is again an open neighbourhood of p, which is the smallest one. It is called the *minimal neighbourhood* of p. The topology of a finite space is completely determined by its minimal neighbourhoods. However, in a general framework of all topological spaces this is not true. Nevertheless, the sets which are realized as arbitrary intersection of open sets in topology are quite interesting. The study of β-open sets and their properties were initiated by Njastad [13] in 1965; his introduction of β -open sets. Andrijevic [17] gave some properties of β-closure of a set A is denoted by βCl(A), and defined as intersection of all β-closed sets containing the set A.

F. Nakaok and N. Oda [19] and [20] introduced the notation of maximal open sets and minimal open sets in topological spaces. In (2010) Mlguel Caldas, Saeid Jafari and Seithuti P. Moshokes [18]; introduce the notion of maximal θ-open, minimal θ-closed, θ-semi maximal open and θ-semi minimal closed and investigate some of the fundamental properties.

In this paper, we have made an investigation of all these type of sets. The minimal open sets, as we call them, being a weaker form of open sets, are studied here in the light of other generalized form of open sets. And the concept of a new class of open sets called maximal β-open sets and minimal β -closed sets. We also investigate some of their fundamental properties

II. PRELIMINARIES

2.1 Definition

Let (X,τ) be a topological space. Then a subset A of (X,τ) is called,

I.Semi-open [10] if $A \subseteq cl\ int\ (A)$.

II.α-open [13] if $A \subseteq int\ cl\ int\ (A)$.

III.Pre-open [11] if $A \subseteq int\ cl\ (A)$.

IV.β-open [1] if $A \subseteq cl\ int\ cl\ (A)$.

V.Regular open (regular closed resp.,)[5]$if\ A = int\ cl(A)\ \ (A = cl\ int(A) resp.,)$

The complement of a semi-open (resp. α-open, pre-open, β open) set is known as semi-closed (resp.α-closed, pre-closed, β-closed) set.

2.2 Definition

A subset S of a topological spaces (X,τ) is said to be

I.An A-set[14] if $S = U \cap C$, where U is open and C is regular closed.

II.A t-set [15] if $int\ (clS) = intS$.

III.A B-set [15] if there is an open set U and a t-set A in X such that $S = U \cap A$.

Let $f: X \to Y$ be a mapping , V be an arbitrary open set in Y. Then f is said to besemi-continuous[10] (resp. pre- continuous[11], α-continuous[12], β-continuous[14]). If $f^{-1}(V)$ is semi-open (resp. pre-open, α-open, β-open) in X. f is said to be A-continuous [14] (resp. B-continuous[15]) if $f^{-1}(V)$ is an A-set (resp. B-sets) in X whenever V is open in Y. It is known that α-continuity implies pre-continuity and semi-continuity, A-continuity implies semi-continuity[14]. It can be shown that a subset S in X is open if and only if it is an A-set and an α-set [14] or equivalently, it is pre-open set and B-set[15]

2.3. [13] Definition

A subset A of a space X is said to be β-open set if $A \subseteq Cl(Int(Cl(A)))$. The complement of all β-open set is said to be β-closed. As in the usual sense, the intersection of all β-closed sets of X containing A is called the β -closure of A. also the union of all β-open sets of X contained in A is called the β-interior of A.

2.4. [21] Definition

A subset A of a space X is said to be θ-open set if for each $x \in A$, there exists an open set G such that $x \in G \subseteq Cl(G) \subseteq A$

2.5. [20] Definition

A proper nonempty open set U of X is said to be a maximal open set if any open set which contains U is X or U.

2.6. [20] Definition

A proper nonempty open set U of X is said to be a minimal open set if any open set which contained in U is ϕ or U.

2.7. [19] Definition

A proper nonempty closed subset F of X is said to be a maximal closed set if any closed set which contains F is X or F.

2.8. [19] Definition

A proper nonempty closed subset F of X is said to be a minimal closed set if any closed set which contained in F is ϕ or F.

2.9. [18] Definition

A proper nonempty θ-open set U of X is said to be a maximal θ-open set if any θ-open set which contains U is X or U.

2.10. [18] Definition

A proper nonempty θ-closed set B of X is said to be a minimal θ-closed set if any θ-closed set which contained in B is ϕ or F.

III. MINIMAL OPEN SETS

In this section, first we define minimal open sets in a topology. It is shown that although, minimal open sets are weaker form of open sets of the given topology, yet they also form a topology on their own.

3.1 Definition

Let (X,τ) be a topological space . A Set $A \subseteq X$ is called minimal open if A can be expressed as intersection of a subfamily of open sets.

The collection of minimal open sets of topology (X,τ) is denoted by M. clearly, every open sets is minimal open. In a finite space, open sets are the only minimal open sets.

The following example gives an idea about the abundance of minimal open sets.

3.2 Example

Let $X = N$, the set of natural numbers, equppied with the cofinite topology. Then every subset of X is minimal open.

3.3 Result

For a topological space (X,τ)

I. $\emptyset, X \in M$
II. M is closed under arbitrary union,
III. M is closed under arbitrary intersection.

Proof

i),iii),are obvious.

ii) hold in view of the fact that P(X), the power set of X forms a completely distributive lattice under union and intersection of sets.

3.4 Definition

Let (X,τ) be a topological space and $A \subseteq X$. Then minimal cover of A, denoted by $C_m(A)$, is defined as $C_m(A) = \bigcap\{U: U \in \tau, A \subseteq U\}$.

From the definition, it follows that $C_m(A)$ is the smallest minimal open set containing A.

3.5 Theorem

Let (X,τ) be a topological space and A,B be subsets of X . Then the following hold:

I. $A \subset C_m(A)$;
II. $C_m(C_m(A)) = C_m(A)$;
III. If $A \subseteq B$ then $C_m(A) \subseteq C_m(B)$;
IV. $C_m(A \cup B) = C_m(A) \cup C_m(B)$;
V. $C_m(\emptyset) = \emptyset$.

Proof

Obvious from the definition of minimal cover of A.

This shows that $C_m(A)$ is a closure operator. An operator similar to C_m was defined for generalized topological spaces in [3]. However the definition in [3] seems to erroneous or incomplete.

3.6 Theorem

Let (X,τ) be a topological spaces. We define

$$\mathfrak{T} = \{A \subseteq X: C_m(A) = A.$$

Then $(X,\mathfrak{T})$ is a topological space and $\tau \subseteq \mathfrak{T}$.

Proof

Clearly $\emptyset, X \in \mathfrak{T}$. Let $A_i \in \mathfrak{T}$ where $\in \wedge$, $A = \bigcup_{i\in\wedge} A_i$. Since $A_i \subseteq \bigcup_{i\in\wedge} A_i$, thus $C_m(A_i) \subseteq C_m(\bigcup_{i\in\wedge} A_i)$ and hence $\bigcup_{i\in\wedge} C_m(A_i) \subseteq C_m(\bigcup_{i\in\wedge} A_i)$. conversely, suppose that $x \notin \bigcup_{i\in\wedge} C_m(A_i)$. Then $x \notin C_m(A_i)$. For each $i \in \wedge$ and hence there exist an open set U_i containing A_i for each i, such that $x \notin U_i$. therefore $x \notin \bigcup_{i\in\wedge} U_i$, which contains $\bigcup_{i\in\wedge} A_i$. Hence $x \notin C_m(\bigcup_{i\in\wedge} A_i)$.

Therefore $\bigcup_{i\in\wedge} C_m(A_i) = C_m(\bigcup_{i\in\wedge} A_i)$. hence $\bigcup_{i\in\wedge} A_i = \bigcup_{i\in\wedge} C_m(A_i) = C_m(\bigcup_{i\in\wedge} A_i) \supseteq \bigcup_{i\in\wedge} A_i$. Lastly we show that if $A, B \in \mathfrak{T}$ then $A \cap B \in \mathfrak{T}$. we know that $C_m(A \cap B) \subseteq C_m(A) \cap C_m(B) = A \cap B$. But $A \cap B \subseteq C_m(A \cap B)$ and hence $C_m(A \cap B) = A \cap B$. Thus $\mathfrak{T}$ form a topology.Furthermore let $U \in \tau$, then $C_m(U) = U$. Thus $U \in \mathfrak{T}$. Hence $\tau \subseteq \mathfrak{T}$.

3.7 Theorem

Let (X,τ) be a topological space. Then $(X,\mathfrak{T})$ form an Alexandroff space[2], that is, it is closed under arbitrary intersection also.

Proof

Let $A_i \in \mathfrak{T}$ for each $i \in\wedge$ then $C_m(\bigcap_{i\in\wedge} A_i) \subseteq C_m(A_i) = A_i$ for each i. Thus $C_m(\bigcap_{i\in\wedge} A_i) \subseteq \bigcap_{i\in\wedge} A_i$. Again $\bigcap_{i\in\wedge} A_i \subseteq C_m(\bigcap_{i\in\wedge} A_i)$. Hence $C_m(\bigcap_{i\in\wedge} A_i) = \bigcap_{i\in\wedge} A_i$. therefore if $A_i \in \mathfrak{T}$ then $\bigcap_{i\in\wedge} A_i \in \mathfrak{T}$.

From **theorem 3.5** and **theorem 3.7** , one can observe that $\mathfrak{T}^c = \{A^c | A \subseteq X: C_m(A) = A\}$ is also a topology.

The members of $\mathfrak{T}$ are called the minimal open or minimal open sets of τ. If X is finite, then $\tau = \mathfrak{T}$.

In the following, we study the interrelationship of minimal open sets with other existing notions and finally obtain a decomposition of open set.

IV A DECOMPOSITION OF OPEN SETS.

The operator C_m defined in the previous section can be used to define new weaker forms of open sets. We show that these weaker forms provide decomposition of open sets as well as that of continuity.

4.1 Definition

Let (X, τ) be a topological space. A subset $S \subseteq X$ is said to be

I. C_m-pre-open if $S \subseteq int(C_m(S))$,

II. C_m-t-set if $int(S) = int(C_m(S))$,

Where int is the interior operator.

One can observe that every open set is C_m-pre open set as well as C_m-t-set. But converse is not true in general. In fact, we have the following example:

4.2 Example

Let $X = \mathbb{R}$, the set of real number with the usual topology. Take $A = (5,6) \cap \mathbb{Q}$. Then A is neither open nor semi-open and α-open set. But A is C_m-pre open-set as $C_m((5,6) \cap \mathbb{Q} = (5,6)$.

Similarly if we take $B = (5,6]$, then B is C_m-t-set but not open.

We can observe that if X is finite, then the class of C_m-preopen sets always forms a discrete topology. Because if X is finite then C_m (A) is an open set containing A.

4.3 Remark

A closed set need not be a C_m-t-set. We have the following example:

4.4 Example

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. If we take $A = \{d\}$. Then A is closed but not C_m-t-set.

In **example 4.4**, we see that $A = \{d\}$ is a t-set but not C_m-t-set whereas $B = \{a, b, c\}$, being an open set is a C_m-t-set, but not a t-set. Thus we can conclude that C_m-t-set is independent of t-set.

Our next example establish a fact that a C_m-t-set is independent of open, semi-open, pre-open, α-open and β-open sets.

4.5 Example

Let $X = \mathbb{R}$, the set of real number with the usual topology. Take $A = (5,6) \cup \{\pi\}$. Then A is neither semi-open,pre-open,α-open and β-open sets. but A is C_m-t-set.

4.6 Example

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. If we take $A = \{a, b, d\}$. Then A is α-open set and hence pre-open, semi-open and β-open set. But A is not C_m-t-set because $C_m(A) = X$.

Also a C_m-t-set independent from A-set and B-set. We have the following example:

4.7 Example

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. If we take $A = \{b, c, d\}$, then A is B-set because A is closed. Burt A is not a C_m-t-set.

4.8 Example

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$. If we take $A = \{c, d\}$, then A is A-set. But A is not a C_m-t-set.

4.9 Example

Let $X = \mathbb{R}$, the set of real numbers with the usual topology. Take $B = [0,1) \cup (1,2) \cup \{\pi\}$. Then B is C_m-t-set. But not a B-set. If we take $A = (5,6) \cup \{\pi\}$ is a C_m-t-set but not an A-set because A is not a semi-open set.

Hence a C_m-t-set is independent from A-set and B-set.

4.10 Proposition

If A,B are two C_m-t-set, then $A \cap B$ is also a C_m-t-set.

Proof

Let A,B be two C_m-t-set. Then $t(A \cap B) \subseteq int(C_m(A \cap B)) \subseteq int\left(C_m(A) \cap C_m(B)\right) = int\left(C_m(A)\right) \cap int\left(C_m(B)\right) = int\,(A) \cap int(B) = int(A \cap B)$. Therefore $int(A \cap B) = int(C_m(A \cap B))$. Hence $A \cap B$is C_m-t-set.

Thus the family of C_m-t-set forms an infratopology [7], where an infrotopology [7] on a set X is collection τ of subsets of X having the following properties:

I. ∅ and X are in τ.

II. The intersection of the element of any finite sub collection of τ is in τ.

In our next theorem, we provide a decomposition of open set in term of C_m-pre-open and C_m-t-set.

4.11 Theorem

Let (X, τ) be a topological space. A subset $S \subseteq X$ is open if and only if C_m-pre-open and C_m-t-set.

Proof

Let $S \subseteq X$ is open set. Therefor S is C_m-pre-open as well as C_m-t-set.

Conversely, let S be a C_m-pre-open and C_m-t-set. We have $S \subseteq int\left(C_m(S)\right) = int(S) \subseteq S$. Hence S is open.

Now we proceed to provide a decomposition for continuous mappings.

4.12 Definition

Let $f: X \to Y$ be a mapping. Then f is said to be

I. C_m-pre-continuous if $f^{-1}(V)\ is\ C_m$-pre-open,

II. C_m-t-continuous if $f^{-1}(V)$ is C_m-t-set, Where V is any arbitrary open set in Y.

From the discussion provide above, it follow that C_m-t-contiuity does not imply semi-continuity, hence does not imply continuity,α-continuity or A-continuity. We have the following example in this regard.

4.13 Example

Let $X = \mathbb{R}$, the set of real number with the usual topology and $Y = \{a, b\}$ with the topology $\mu = \{\emptyset, \{a\}, Y\}$. Let $f: X \to Y$ be defined as:

$$f(x) = a \quad where\ A = (5,6) \cap \mathbb{Q}$$
$$= b\ if\ x \notin A$$

Then f is C_m -t-continuous but neither semi-continuous nor continuous.

C_m-t-continuity is independent from B-continuity and A-continuity also. Here are the example:

4.14 Example

Let $X = \mathbb{R}$, the set of real number with the usual topology and $Y = \{a, b\}$ with the topology $\mu = \{\emptyset, \{a\}, Y\}$. Let $f: X \to Y$ be defined as:

$$f(x) = a\ where\ A = [5,6) \cup (6,7) \cup \{\pi\}$$
$$= b\ if\ x \notin A$$

Then f is C_m-t-continuous but not B-continuous.

4.15 Example

Let $X = \{a, b, c, d\}$, with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $Y = \{x, y\}$ with the topology $\mu = \{\emptyset, \{x\}, Y\}$. Let $f: X \to Y$ be defined as: $f(a) = y, f(b) = f(c) = f(d) = x$. Then f is B-continuous but not C_m-t-continuous.

Similarly,

4.16 Example

Let $X = \{a, b, c, d\}$, with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ on X and topology $\mu = \{\emptyset, \{c\}, \{c, d\}, \{a, b, c\}, Y\}$. let $f: X \to Y$ be defined as identity map. Then f is A-continuous but not C_m-t-continuous because $\{c, d\}$ is not a C_m-t-set in X.

From **theorem 4.11** we have decomposition of continuity in the following manner:

4.17 Theorem

A mapping $f: X \to Y$is continuous if and only if it is both C_m-pre continuous and C_m-t-continuous.

V. MINIMAL SEPARATION AXIOMS:

In this section we defied and introduced minimal open sets through give the relation between minimal T_0, minimal T_1, minimal T_2, minimal T_3 and minimal T_4 and also examples.

5.1 Definition

A space X is a minimal-T_0 space iff it satisfies the T_0 axiom, i.e., for each $x, y \in X$ such that $x \neq y$ there is an minimal open set $U \subset X$ so that U contains one of x and y but not the other.

5.2 Definition

A space X is a minimal T_1 space iff it satisfies the T_1axiom, i.e., for each $x, y \in X$ such that $x \neq y$ there is an minimal open set $U \subset X$ so that $x \in U$ but $y \notin U$.

5.3 Example

The set $\{0,1\}$ furnished with the topology $\{\emptyset, \{0\}, \{0,1\}\}$ is called sierpinski space. It is minimal-T_0 but not minimal-T_1.

5.4 Remark

Every T_0space is minimal -T_1space but converse is not true in general.

5.5 Definition

A space X is a minimal-T_2space or minimal hausdorff space iff it satisfies the T_2axiom, i.e., for each $x, y \in X$such that $x \neq y$ there are minimal open sets $U, V \subset X$ so that $x \in U, y \in V$ and $U \cap V = \emptyset$.

5.6 Remark

Every minimal-T_1 space is minimal-T_2 space but converse is not true in general.

5.7 Definition

A space X is minimal regular iff for each $x \in X$ and each minimal closed $C \subset X$ such that $x \notin C$ there are minimal open sets $U, V \subset X$ so that $x \in U, C \subset V$ and $U \cap V = \emptyset$. A regular minimal-T_1 space is called a minimal-T_3.

5.8 Remark

Every T_0 space minimal regular . But converse is not true in general.

5.9 Remark

Every minimal hausdrorff space is minimal regular. But converse is not true in general.

5.10 Definition

A space X is minimal normal iff for each pair A, B of disjoint minimal closed subsets of X,there is a pair U, V disjoint minimal open subsets of X so that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. A minimal normal T_1 space is called a minimal T_4 space.

5.11 Remark

Every regular space is minimal normal but converse is not true in general.

VI. MINIMAL REGULARITY

In this section, we further use the concept of minimal open sets to define a weaker form of regularity. Some interesting results are obtained here. While Hausdorffness implies minimal regularity. Also the product of normal spaces is found to be minimal normal.

6.1 Definition

A topological space X is said to be minimal regular if for each pair consist of a point x and a closed set B not containing x, there exist a disjoint pair of minimal open set and an open set, containing x and B respectively.

Since every open set is minimal open therefore every regular space is minimal regular as well. Converse is however not true. Here is an example.

6.2 Example

Let $X = \mathbb{N}$, the set of natural numbers, equipped with the cofinite topology. Then every subset of X is minimal open. Then X is a minimal regular space but not a regular space.

6.3 Theorem

Let X be a topological space then X is minimal regular if and only if given a point $x \in X$ and an open neighborhood U of x, there exists an minimal open neighborhood V of x such that $x \in V \subseteq cl(V) \subseteq U$.

Proof

Suppose that X is minimal regular space and x and an open neighborhood U of x are give. Then $B = X \backslash U$, is a closed set not containing x. By the hypothesis, there exists a pair of disjoin minimal open set V and open set W, containing x and B respectively, that is, $x \in V$ and $B \subseteq W$. then $cl(V)$ is disjoint from B and hence $x \in V \subseteq cl(V) \subseteq X \backslash W \subseteq X \backslash B \subseteq U$, then is $cl(V) \subseteq U$.

Conversely, suppose that a point x and a closed set B not containing x are given. Then $U = X \backslash B$, is an open set containing x. Therefore there is a minimal open neighborhood V of x such that $cl(V) \subseteq U$. Then the minimal open sets V and the open set $X \backslash cl(V)$, are disjoint sets containing x and B respectively. Thus (X, τ) is minimal regular.

6.4 Theorem

Every Hausdorff space is minimal regular.

Proof

Let X be a hausdorff space. Let x and B be a pair of a point and a closed set not containing the point x. then for every $y \in B$, we have $x \neq y$. Therefore by the given hypothesis, there exist disjoint open set U_y and V_y such that $\in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Then $\{V_y | y \in B\}$ is an open cover of B and $U = \cap_{y \in B} U_y$, is an minimal open set containing x such that $U \cap V = \emptyset$. Therefore X is minimal regular.

Our next theorem is on the product of regular space.

6.5 Theorem

Product of minimal regular spaces is again minimal regular.

Proof

Let $\{X_\alpha\}$ be a family of minimal regular spaces. Let $X = \prod_\alpha X_\alpha$. Let $x = \{x_\alpha\}$ be a point of X and U be an open neighbourhood of $x \in X$. We choose a basis element $\prod_\alpha X_\alpha$ about x contained in U. then $x_\alpha \in U_\alpha$ for each α . As X_α is minimal regular, there exists an minimal open set V_α in X_α such that $x_\alpha \in V_\alpha \subseteq cl(V_\alpha) \subseteq U_\alpha$. Now , V_α being minimal open, we have, $V_\alpha = \cap_j V\alpha_j$, where $V\alpha_j$ is open in X_α. If $U_\alpha = X_\alpha$, we simply choose $V_\alpha = X_\alpha$. Then by using the fact [5] that $\cap_\beta \left(\prod_\alpha A_{\alpha,\beta}\right) = \prod_\alpha (\cap_\beta A_{\alpha,\beta})$, we find that $V = \prod_\alpha V_\alpha$ is an minimal open set in $\prod_\alpha X_\alpha$. Since $cl(V) = cl(\prod_\alpha V_\alpha) = \prod_\alpha cl(V_\alpha)$, it follows that $x \in V \subseteq cl(V) \subseteq \prod_\alpha U_\alpha \subseteq U$. hence X is minimal regular.

6.6 Definition

A topological space X is said to be minimal normal if every pair of disjoint closed sets are contained in disjoint minimal open sets one can observe that every normal space is minimal normal. But converse is not true in general.

In **Example 6.2** X is minimal normal but not normal.

In our next theorem, we provide that minimal regular spaces are minimal normal.

6.7 Theorem

Every minimal regular space is minimal normal.

Proof

Let X be minimal regular space. Let A and B be disjoint closed subsets of X. then for each $x \in A$ has an open neighbourhood $X \backslash B$, because $A \subseteq X \backslash B$. By the given hypothesis, there exists an minimal open set V_x such that $cl(V_x)$ doesn't intersect B, that is $x \in V_x \subseteq cl(V_x) \subseteq X \backslash B$. therefore $\{V_x | x \in A\}$ forms an minimal open covering of A. in the same way,$\{ U_x | x \in B\}$ is also an minimal open covering of B. thus $V = \cup_{a \in A} V_a$ and $U = \cup_{b \in B} U_b$ are minimal open sets containing A and B respectively.

Now we define $U_i' = U_i \backslash \cup_{a \in A, a \neq i} cl(V_a)$ and $V_i' = V_i \backslash \cup_{b \in \Lambda', b \neq i} cl(U_b)$. here U_i' and V_i' are minimal open sets. Since arbitrary union of closed sets are minimal closed, therefore $\cup_{a \in A, a \neq i} cl(V_a)$ is minimal closed. (Because if A is minimal open, then $A = \cap_n \{V_n \text{ , where} V_n \text{ is open}\}$. thus $X \backslash A = X \backslash \cap_n V_n = \cup_n (X \backslash V_n) = \cup_n U_n$, where $U_n = X \backslash V_n$, closed sets). Thus $U_i \backslash \cup_{a \in A, a \neq i} cl(V_a) = U_i \cap [\cup_{a \in A, a \neq i} cl(V_a)]^c$. here $[\cup_{a \in A, a \neq i} cl(V_a)]^c$ is an minimal open set in X, therefore $U_i' = U_i \backslash \cup_{a \in A, a \neq i} cl(V_a)$ is again an minimal open set in X. similarly V_i' is minimal open. Now, we have $U' = \cup U_i'$ and $V' = \cup V_i'$ are disjoint minimal open covers containing A and B respectively. Therefore X is minimal normal space.

6.8 Corollary

Thus from the **theorem 6.7** and **6.5** we can say that the product of normal T_1-space is coming out be minimal normal.

VII. MAXIMAL AND MINIMAL β-OPEN SETS.

7.1 Definition

A proper nonempty β-open set A of X is said to be a maximal β-open set if any β-open set which contains A is X or A.

7.2 Definition

A proper nonempty β-closed set B of X is said to be a minimal β -closed set if any β -closed set which contained in B is ϕ or B.

The family of all maximal β-open (resp.; minimal β -closed) sets will be denoted by
$M_a\beta O(X)(resp.; M_i\beta C(X)$. we set
$M_a\beta O(X,x) = \{A: x \in A \in M_a\beta O(X)\}$,and
$M_i\beta C(X,x) = \{F: x \in \text{family } A \in M_i\beta C(X,x)\}$.

7.3 Theorem

Let A be a proper nonempty subset of X. Then A is a maximal β-open set if $X \backslash A$ is a minimal β-closed set.

Proof

Necessity. Let A be a maximal β-open. Then $A \subset X$ or $A \subset A$. Hence,$\emptyset \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Therefore, by Definition 7.2, $X\backslash A$ is a minimal β -closed set.

Sufficiency. Let $X\backslash A$ is be a minimal β-closed set. Then $\emptyset \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Hence, $A \subset X$ or $A \subset A$ which implies that A is a maximal β-open set.

The following example shows that maximal-open sets and maximal β-open sets are in general independent.

7.4 Example

Consider $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. Then the family of $\beta O(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. So $\{a\}$ is a maximal open in X, which is not maximal β-open in X, and $\{a, b\}$ is a maximal β-open in X, which is not maximal open in X.

7.5 Theorem

Any open set if it is a maximal β-open set then it is a maximal open set.

Proof

Let U be open and maximal β-open set in a topological space X. We want to prove that U is a maximal open set. Suppose that U is not maximal open set, then $U \neq X$ and there exists an open set G such that $U \subset G$ and $U \neq G$, but every open set is β-open, this implies that G is a β-open set containing U and $U \neq G$ and $U \neq X$, which is contradiction. Hence U is a maximal open set.

7.6 Theorem

For any topological space X, the following statements are true.

1) Let A be a maximal β -open set and B be a β-open. Then $A \cup B = X \; or \; B \subset A$.
2) Let A and B be maximal β-open sets. Then $A \cup B = X \; or \; B = A$.
3) Let F be a minimal β-closed set and G be a β-closed set. Then $F \cap G = \emptyset \; or \; F \subset G$.
4) Let F and G be minimal β-closed sets. Then $F \cap G = \emptyset \; or \; F = G$.

Proof (1)

Let A be a maximal β-open set and B be a β-open set. If $A \cup B = X$, then we are done. But if $A \cup B \neq X$, then we have to prove that $B \subset A$, but $A \cup B \neq X$ means $B \subset A \cup B$ and $A \subset A \cup B$. Therefore we have $A \subset A \cup B$ and A is a maximal β-open, then by Definition 7.1, $A \cup B = X$ or $A \cup B = A$, but $A \cup B \neq X$,then $A \cup B = A$, which implies $B \subset A$.

Proof (2)

Let A and B be maximal β-open sets. If $A \cup B = X$, then we have done. But if $A \cup B \neq X$, then we have to prove that $B = A$. Now $A \cup B \neq X$, means $B \subset A \cup B$ and $A \subset A \cup B$. Now $A \subset A \cup B$ and A is a maximal β -open, then by Definition 7.1, $A \cup B = X$ or $\cup B = A$, but $A \cup B \neq X$, therefore, $A \cup B = A$, which implies $B \subset A$. Similarly if $B \subset A \cup B$ we obtain $A \subset B$. Therefore $B = A$.

Proof (3)

Let F be a minimal β -closed set and G be a β -closed set. If $F \cap G = \emptyset$, then there is nothing to prove. But if $F \cap G \neq \emptyset$, then we have to prove that $F \subset G$. Now if $F \cap G \neq \emptyset$, then $F \cap G \subset F$, and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal β-closed, then Definition 7.2, $F \cap G = F$ or $F \cap G = \emptyset$. But $F \cap G \neq \emptyset$, then $F \cap G = F$, which implies $F \subset G$.

Proof (4)

Let F and G be two minimal β-closed sets. If $F \cap G = \emptyset$, then there is nothing to prove. But if $F \cap G \neq \emptyset$,then we have to prove that $F = G$. Now if $F \cap G \neq \emptyset$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal β-closed, then by Definition 7.2, $F \cap G = F$ or $F \cap G = \emptyset$. But $F \cap G \neq \emptyset$ then $F \cap G = F$, which implies $F \subset G$. Similarly if $F \cap G \subset G$ and given that G is minimal β-closed, then by Definition 7.2, $F \cap G = G$ or $F \cap G = \emptyset$ But $F \cap G \neq \emptyset$, then $F \cap G = G$ which implies $G \subset F$. Then $F = G$.

7.7 Theorem

Let A be a maximal β-open set and x is an element of $X\backslash A$. Then $X\backslash A \subset B$ for any β -open set B containing x.

1) Let A be a maximal β-open set. Then either of the following (i) or (ii) holds:
   i. For each $x \in X\backslash A$, and each β-open set B containing x, $B = X$.
   ii. There exists a β-open set B such that $X\backslash A \subset B$, and $B \subset X$.
2) Let A be a maximal β open set. Then either of the following (i) or (ii) holds:
   i. For each $x \in X\backslash A$, and each β-open set B containing x, we have $X\backslash A \subset B$.
   ii. There exists a β-open set B such that $X\backslash A = B \neq X$.

Proof

1) Since $x \in X\backslash A$, we have $B \not\subset A$ for any β-open set B containing x. Then $A \cup B = X$, by Theorem 7.5(1). Therefore, $X\backslash A \subset B$.
2) If (i) does not hold, then there exists an element x of $X\backslash A$, and a β -open set B containing x such that $B \subset X$. By (1) we have $X\backslash A \subset B$.
3) If (ii) does not hold, then we have $X\backslash A \subset B$ for each $x \in X\backslash A$ and each β-open set B containing x. Hence, we have $X\backslash A \subset B$.

7.8 Theorem

Let A, B and C be maximal β-open sets such that $A \neq B$. If $A \cap B \subset C$, then either $A = C \; or \; B = C$.

Proof

Given $A \cap B \subset C$. If $A = C$, then there is nothing to prove. But If $A \neq C$, then we have to prove $B = C$. Using Theorem 7.6(2), we have

$$\begin{aligned} B \cap C &= B \cap [C \cap X] \\ &= B \cap [C \cap (A \cup B)] \\ &= B \cap [(C \cap A) \cup (C \cap B)] \\ &= (B \cap C \cap A) \cup (B \cap C \cap B) \\ &= (A \cap B) \cup (C \cap B), since\; A \cap B \subset C \\ &= (A \cup C) \cap B \\ &= X \cap B = B, since\; A \cup C = X. \end{aligned}$$

This implies $B \subset C$ also from the definition of maximal β-open set it follows that $B = C$.

7.9 Theorem

Let A, B and C be maximal β -open sets which are different from each ether. Then $(A \cap B) \not\subset (A \cap C)$.

Proof

Let $(A \cap B) \subset (A \cap C)$. Then $(A \cap B) \cup (C \cap B) \subset (A \cap C) \cup (C \cap B)$. Hence, $(A \cup C) \cap B \subset C \cap (A \cup B)$. Since by Theorem 7.6(2), $A \cup C = X$. We have $X \cap B \subset C \cap X$ which implies $B \subset C$. From the definition of maximal β-open set it follows that $B = C$. Contradiction to the fact that A, B and C are different from each other. Therefore $(A \cap B) \not\subset (A \cap C)$.

7.10 Theorem

1) Let F be a minimal β-closed set of X. If $x \in F$, then $F \subset G$ for any β-closed set G containing x.
2) Let F be a minimal β-closed set of X. Then $F = \cap \{G: G \in \beta C(X)\}$.

Proof

1) Let $F \in M_i\beta C(X, x)$and $G \in \beta C(X, x)$, such that $F \not\subset G$. This implies that $F \cap G \subset F$ and $F \cap G \neq \emptyset$. But since F is minimal β-closed, by Definition 2.4, $F \cap G = F$ which contradicts the relation that $F \cap G \subset F$. Therefore $\subset G$.
2) By (1) and the fact that F is β-closed containing x, we have $\subset \cap \{G: G \in \beta C(X)\} \subset F$. Therefore we have the result.

7.11 Theorem

1) Let F and $\{F_\lambda\}_{\lambda\in\Lambda}$ be minimal β-closed sets. If $\subset \bigcup_{\lambda\in\Lambda} F_\lambda$, then there exist $\lambda \in \Lambda$ such that $F = F_\lambda$.
2) Let F and $\{F_\lambda\}_{\lambda\in\Lambda}$ be minimal β-closed sets. If $F \neq F_\lambda$ for each $\lambda \in \wedge$, then $\bigcup_{\lambda\in\Lambda} F_\lambda \cap F = \emptyset$

Proof

1) Let F and $\{F_\lambda\}_{\lambda\in\Lambda}$be minimal β-closed sets with $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda$ .we have to prove that $F \cap F_\lambda \neq \emptyset$. Since if $F \cap F_\lambda = \emptyset$, then $F_\lambda \subset X\backslash F$ and hence, $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda \subset X\backslash F$ which is a contradiction. Now as $F \cap F_\lambda \neq \emptyset$, then $F \cap F_\lambda \subset F$ and $F \cap F_\lambda \subset F_\lambda$. Since $F \cap F_\lambda \subset F$ and give that F is minimal β-closed , then by Definition 2.3, $F \cap F_\lambda = F$ or $F \cap F_\lambda = \emptyset$. But $F \cap F_\lambda \neq \emptyset$, then $F \cap F_\lambda = F$ which implies $F \subset F_\lambda$. Therefore, $F = F_\lambda$.
2) Suppose that $(\bigcup_{\lambda\in\Lambda} F_\lambda) \cap F \neq \emptyset$ then there exists $\lambda \in \Lambda$ such that $F \cap F_\lambda \neq \emptyset$. By Theorem 7.6(4), we have $F_\lambda = F$ which is a contradiction to the fact $F \neq F_\lambda$. Hence, $(\bigcup_{\lambda\in\Lambda} F_\lambda) \cap F = \emptyset$.

7.12 Theorem

Let U be a maximal β-open set. Then $\beta Cl(U) = X$ or $\beta Cl(U) = U$.

Proof

Since U is a maximal β-open sets, the only following cases (1) and (2) occur by theorem 7.7(2):

1) For each $x \in X\backslash U$ and each β-open set W of x, we have $X\backslash U \subset W$, let x be any element of $X\backslash U$ and W be any β-open set of x. since $X\backslash U \neq W$, we have $W \cap U \neq \emptyset$ for any β-open set W of x. hence, $X\backslash U \subset \beta Cl(U)$. Since $X = X \cup (X\backslash U) \subset U \cup \beta Cl(U) = \beta Cl(U) \subset X, we\ have\ \beta Cl(U) = X$.
2) There exists a β-open set W such that $X\backslash U = W \neq X$, since $X\backslash U = W$ is a β -open set, U is a β-closed set. Therefore, $U = \beta Cl(U)$.

7.13 Theorem

Let U be a maximal β-open set. Then $\beta Int(X\backslash U) = X - U$ or $\beta Int(X\backslash U) = \emptyset$.

Proof

By Theorem 7.7, we have following cases (1) $\beta Int(X\backslash U) = \emptyset$ or(2) $\beta Int(X\backslash U) = X\backslash U$.

7.14 Theorem

Let U be a maximal β-open set and S a nonempty subset of $X\backslash U$. Then $\beta Cl(S) = X\backslash U$.

Proof

Since $\emptyset \neq S \subset X\backslash U$, we have $W \cap S \neq \emptyset$. for any element x of $X\backslash U$and any β -open set W of x by Theorem 7.12. Then $X\backslash U \subset \beta Cl(U)$. Since $X\backslash U$ is a β-closed set and $S \subset X\backslash U$, we see that $\beta Cl(S) \subset \beta Cl(X\backslash U) = X\backslash U$. therefore $X\backslash U = \beta Cl(S)$

7.15 Corollary

Let U be a maximal β-open set and M a subset of X with $U \subset M$. Then $\beta Cl(M) = X$.

Proof

Since $U \subset M \subset X$,there exists a nonempty subset S of $X\backslash U$ such that $M = U \cup S$. Hence we have $\beta Cl(M) = \beta Cl(U \cup S) = \beta Cl(U) \cup \beta Cl(S) \supset (X\backslash U) \cup U = X$ by Theorem 7.14. Therefore $\beta Cl(M) = X$.

7.16 Theorem

Let U be a maximal β-open set and assume that the subset $X\backslash U$ has two element at least. Then $\beta Cl(X\backslash\{a\}) = X$ for any element of $X\backslash U$.

Proof

Since $U \subset X\backslash\{a\}$ by our assumption, we have the result by Corollary 7.15.

7.17 Theorem

Let U be a maximal β-open set, and N be a proper subset of X with $U \subset N$. Then, $\beta Int(N) = U$.

Proof

If $N = U$, then $\beta Int(N) = \beta Int(U) = U$. Otherwise, $N \neq U$, and hence $U \subset N$. It follows that $U \subset \beta Int(N)$. Since U is a maximal β-open set, we have also $\beta Int(N) \subset U$. Therefore $\beta Int(N) = U$.

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