THE STABILITY OF CUBIC AND QUARTIC FUNCTIONAL EQUATION IN F-SPACES

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ABSTRACT. The intention of this paper is to know some result for the stability of cubic and quartic functional equation in F-Spaces.

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MSC: 39B52; 39B72; 39B82; 46L07; 47L25

1 INTRODUCTION AND PRELIMINARIES

The Stability Problem for functional equations starts from the famous talk of Ulam and the partial solution of Hyers to the Ulam's problem [2, 9]. Thereafter, Rassias [8] attempted to solve the stability Problem of the Cauchy additive functional equation in a more general setting. The concept introduced by Rassias's theorem significantly influenced a number of Mathematicians to investigated the stability problems for various functional equations see [1, 2, 3, 5, 7]. Recently, Xiuzhoung Yang [10] proved the Hyers Ulam-Rassias stability of quadratic functional equation and Orthogonal stability of the Pexiderized quadratic functional equation in F-Spaces. In [4], Jun and Kim considered the following cubic functional equation

$$f(2u+v)+f(2u-v)=2f(u+v)+2f(u-v)+12f(u).$$

It is easy to show that the equation $f(u)=u^3$ satisfies the above functional equation, which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping. In [6], Lee et al. considered the following quartic functional equation

$$f(2u+v)+f(2u-v)=4f(u+v)+4f(u-v)+24f(u)-6f(v).$$

It is easy to show that the function $f(u)=u^4$ satisfies the above functional equation, which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping. We recall some basic facts concerning F-spaces.

Definition 1.1. [10] Let U be a linear space over **K** that denotes either complex or real numbers. A nonnegative valued function $\|.\|$ defined on U is called F-norm if it satisfies the following conditions:

(1) $\|u\|=0$ if and only iff $u=0.$

(2) $\|xu\|=\|u\|$ for all $x\in K, |x|=1.$

(3) $\|u+v\|\le\|u\|+\|v\|.$

(4) $\|x_n u\|\to 0$ provided $x_n\to 0.$

(5) $\|xu_n\|\to 0$ provided $u_n\to 0.$

(6) $\|x_n u_n\|\to 0$ provided $x_n\to 0,\ u_n\to 0.$

A linear space equipped with F-norm is called F^*-space which will be denoted by $(U,\|.\|)$ or U. A complete F^*-space is called F-space.

Definition 1.2. [10] Let U be a linear space over **K** that denotes either complex or real numbers and $0<\beta\le 1.$ A nonnegative valued function $\|.\|$ defined on U is called β - norm if it satisfies the following conditions:

(1) $\|u\|=0$ if and only iff u=0.

(2) $\|xu\|=|x^{\beta}|\|u\|$ for all $x\in \mathrm{K}.$

(3) $\|u+v\|\le\|u\|+\|v\|.$

A linear space equipped with β - norm is called β - normed space, which will be denoted by $(U,\|.\|)$ or briefly U.

In this paper, we present the Stability of Cubic and Quartic Functional Equations in F-Spaces.

2 THE STABILITY OF CUBIC FUNCTIONAL EQUATION IN F-SPACES

Theorem 2.1. Let U be a real vector space and V be F-space in which there exist $\frac{1}{2} \le c < 1$ such that $\left\|\frac{v}{2}\right\| \le c\|v\|$ for all $u \in V$, and let $\varphi : U \times U \to R^{+}$ be a function such that

$$\sum_{i=0}^{\infty} c^{3i}\varphi(2^{i}u,0) \qquad (2.1)$$

Converges and $$\lim_{n\to\infty} c^{3n}\varphi(2^{n}u,2^{n}v)=0 \text{ for all } u,v\in U. \qquad (2.2)$$

Suppose that f satisfies

$$\left\|f(2u+v)+f(2u-v)-2f(u+v)-2f(u-v)-12f(u)\right\| \le \varphi(u,v) \text{ for all } u,v\in U. \quad (2.3)$$

Then there exist a unique cubic function $g : U \to V$ which satisfies the following equation

$$f(2u+v)+f(2u-v)-2f(u+v)-2f(u-v)-12f(u)=0 \quad \text{for all } u,v\in U. \qquad (2.4)$$

and the inequality $$\left\|f(u)-g(u)\right\| \le c^{4}\sum_{i=0}^{\infty} c^{3i}\varphi(2^{i}u,0) \qquad \text{for all } u\in U. \qquad (2.5)$$

Function g is given by $$g(u)=\lim_{n\to\infty} 2^{-3n} f(2^{n}u) \quad \text{for all } u,v\in U. \qquad (2.6)$$

Proof:

Put v = 0 in inequality (2.3), we have

$$\left\|2f(2u)-16f(u)\right\| \le \varphi(u,0) \qquad \text{for all } u\in U. \qquad (2.7)$$

Divide by 16, we get

$$\left\|f(u)-8^{-1}f(2u)\right\| \le c^{4}\varphi(u,0) \qquad \text{for all } u\in U. \qquad (2.8)$$

Replace u by 2u and multiply 8^{-1} on both sides of inequality (2.8), we get

$$\left\|8^{-1}f(2u)-8^{-2}f(2^{2}u)\right\| \le c^{4}c^{3}\varphi(2u,0) \qquad \text{for all } u\in U. \qquad (2.9)$$

Now adding the inequality (2.8) and (2.9), we get

$$\left\|f(u)-8^{-2}f(2^{2}u)\right\| \le c^{4}\left[\varphi(u,0)+c^{3}\varphi(2u,0)\right] \text{ for all } u\in U. \quad (2.10)$$

Similarly we have

$$\left\|f(u)-8^{-3}f(2^{3}u)\right\| \le c^{4}\left[\varphi(u,0)+c^{3}\varphi(2u,0)+c^{6}\varphi(2^{2}u,0)\right] \text{ for all } u\in U. \qquad (2.11)$$

By using induction on positive integer n, we have

$$\left\|f(u)-8^{-n}f(2^{n}u)\right\| \le c^{4}\sum_{i=0}^{n-1} c^{3i}\varphi(2^{i}u,0) \quad \text{for all } u\in U. \qquad (2.12)$$

We have to prove the sequence $\left\{8^{-n}f(2^{n}u)\right\}$ is convergence.

Now replace u by $2^{m}u$ and multiply 8^{-m} on both sides of inequality (2.12), we have

$$\left\|8^{-(n+m)}f(2^{n+m}u)-8^{-m}f(2^{m}u)\right\| \le c^{4}\sum_{i=m}^{n} c^{3i}\varphi(2^{i}u,0) \text{ for all } u\in U. \qquad (2.13)$$

The right hand side $\to 0$ as $m\to\infty$

$\Rightarrow$ The sequence $\left\{8^{-n}f(2^{n}u)\right\}$ is a Cauchy sequence.

$\Rightarrow \sum_{i=0}^{\infty} c^{3i}\varphi(2^{i}u,0)$ is converges.

Now define $$g(u)=\lim_{n\to\infty} 2^{-3n} f(2^{n}u) \quad \text{for all } u\in U.$$

Now taking limit on both sides of the inequality (2.12), we have

$$\left\|f(u)-g(u)\right\| \le c^{4}\sum_{i=0}^{\infty} c^{3i}\varphi(2^{i}u,0) \qquad \text{for all } u\in U.$$

Now we have to prove T satisfies the equation (2.4).

Here replace u and v by $2^n u, 2^n v$ respectively and multiply 8^{-n} on both sides of inequality (2.3), we have

$$\left\|8^{-n}\left[f\left(2^n(2u+v)\right)+f\left(2^n(2u-v)\right)-2f\left(2^n(u+v)\right)-2f\left(2^n(u-v)\right)-12f\left(2^n u\right)\right]\right\|\le c^{3n}\varphi\left(2^n u,2^n v\right)$$

for all $u,v\in U.$ (2.14) Taking limit as n $\to\infty$ we find that g satisfies equation (2.4).

$$\left\|g\left(2u+v\right)+g\left(2u-v\right)-2g\left(u+v\right)-2g\left(u-v\right)-12g\left(u\right)\right\|\le \lim_{n\to\infty}c^{3n}\varphi\left(2^n u,2^n v\right) \text{ for all } u,v\in U.$$

$$g\left(2u+v\right)+g\left(2u-v\right)-2g\left(u+v\right)-2g\left(u-v\right)-12g\left(u\right)=0 \quad \text{for all } u,v\in U.$$

Now we have to prove the uniqueness of a cubic function T subject to inequality (2.5). Let us assume that there exist a cubic function S: U→ V which satisfies the equation (2.4) and inequality (2.5).

$S(2^n u)=8^n S(u)$ and $T(2^n u)=8^n T(u)$ for all $u\in U$ and $n\in N.$

$$\Rightarrow \left\|S(u)-T(u)\right\|=\left\|8^{-n}S(2^n u)-8^{-n}T(2^n u)\right\|$$

$$=\left\|8^{-n}\left[S(2^n u)-f(2^n u)\right]+8^{-n}\left[f(2^n u)-T(2^n u)\right]\right\|$$

$$\le c^4\sum_{i=0}^{\infty}c^{3(n+i)}\varphi(2^{n+i}u,0) \quad \text{for all } u\in U .$$

By letting $n\to\infty$ in the proceeding inequality, we find the uniqueness of g.

Hence the proof.

Corollary 2.2. Let U be a real vector space and V be a complete β -normed space $\left(0<\beta\le 1\right)$ and let $\varphi: U\times U\to R^+$ be a function such that

$$\sum_{i=0}^{\infty}8^{-\beta i}\varphi(2^i u,0) \tag{2.15}$$

Converges and $\lim_{n\to\infty}8^{-\beta n}\varphi(2^n u,2^n v)=0$ for all $u,v\in U.$ (2.16)

Suppose that f satisfies

$$\left\|f\left(2u+v\right)+f\left(2u-v\right)-2f\left(u+v\right)-2f\left(u-v\right)-12f\left(u\right)\right\|\le\varphi\left(u,v\right) \text{ for all } u,v\in U. \tag{2.17}$$

Then there exist a unique cubic function $g: U\to V$ which satisfies the following equation

$$f\left(2u+v\right)+f\left(2u-v\right)-2f\left(u+v\right)-2f\left(u-v\right)-12f\left(u\right)=0 \quad \text{for all } u,v\in U. \tag{2.18}$$

and the inequality $\left\|f\left(u\right)-g\left(u\right)\right\|\le 16^{-\beta}\sum_{i=0}^{\infty}8^{-\beta i}\varphi\left(2^i u,0\right)$ for all $u\in U.$ (2.19)

Function g is given by $g\left(u\right)=\lim_{n\to\infty}2^{-3n}f\left(2^n u\right)$ for all $u,v\in U.$ (2.20)

3 THE STABILITY OF QUARTIC FUNCTIONAL EQUATION IN F-SPACES

Theorem 3.1. Let U be a real vector space and V be F-space in which there exist $\frac{1}{2}\le c<1$ such that $\left\|\frac{v}{2}\right\|\le c\left\|v\right\|$ for all $u\in V,$ and let $\varphi: U\times U\to R^+$ be a function such that

$$\sum_{i=0}^{\infty}c^{4i}\varphi(2^i u,0) \tag{3.1}$$

Converges and $\lim_{n\to\infty}c^{4n}\varphi(2^n u,2^n v)=0$ for all $u,v\in U.$ (3.2)

Suppose that f satisfies $\left\|f\left(2u+v\right)+f\left(2u-v\right)-4f\left(u+v\right)-4f\left(u-v\right)-24f\left(u\right)+6f\left(v\right)\right\|\le\varphi\left(u,v\right)$ for all $u,v\in U.$ (3.3)

Then there exist a unique quartic function $g: U\to V$ which satisfies the following equation

$$f\left(2u+v\right)+f\left(2u-v\right)-4f\left(u+v\right)-4f\left(u-v\right)-24f\left(u\right)+6f\left(v\right)=0 \quad \text{for all } u,v\in U. \tag{3.4}$$

and the inequality $\left\|f\left(u\right)-g\left(u\right)\right\|\le c^5\sum_{i=0}^{\infty}c^{4i}\varphi\left(2^i u,0\right)$ for all $u\in U.$ (3.5)

Function g is given by $g\left(u\right)=\lim_{n\to\infty}2^{-4n}f\left(2^n u\right),$ for all $u,v\in U.$ (3.6)

Proof:

Put $v=0$ in inequality (3.3), we have

$$\left\|2f(2u)-32f(u)\right\|\le\varphi(u,0) \quad \text{for all} \quad u\in U. \tag{3.7}$$

Divide by 32, we get

$$\left\|f(u)-16^{-1}f(2u)\right\|\le c^5\varphi(u,0) \quad \text{for all} \quad u\in U. \tag{3.8}$$

Replace u by 2u and multiply 16^{-1} on both sides of inequality (3.8), we get

$$\left\|16^{-1}f(2u)-16^{-2}f(2^2u)\right\|\le c^5c^4\varphi(2u,0) \quad \text{for all} \quad u\in U. \tag{3.9}$$

Now adding inequality (3.8) and (3.9), we get

$$\left\|f(u)-16^{-2}f(2^2u)\right\|\le c^5\left[\varphi(u,0)+c^4\varphi(2u,0)\right] \quad \text{for all} \quad u\in U. \tag{3.10}$$

Similarly we have

$$\left\|f(u)-16^{-3}f(2^3u)\right\|\le c^5\left[\varphi(u,0)+c^4\varphi(2u,0)+c^8\varphi(2^2u,0)\right] \quad \text{for all} \quad u\in U. \tag{3.11}$$

By using induction on positive integer n, we have

$$\left\|f(u)-16^{-n}f(2^nu)\right\|\le c^5\sum_{i=0}^{n-1}c^{4i}\varphi(2^iu,0) \quad \text{for all} \quad u\in U. \tag{3.12}$$

We have to prove the sequence $\left\{16^{-n}f(2^nu)\right\}$ is convergence.

Now replace u by 2^mu and multiply 16^{-m} on both sides of inequality (3.12), we have

$$\left\|16^{-(n+m)}f(2^{n+m}u)-16^{-m}f(2^mu)\right\|\le c^5\sum_{i=m}^{n}c^{4i}\varphi(2^iu,0), \quad \text{for all} \quad u\in U. \tag{3.13}$$

The right hand side $\to 0$ as $m\to\infty$

$\Rightarrow$ The sequence $\left\{16^{-n}f(2^nu)\right\}$ is a Cauchy sequence.

$\Rightarrow$ $\sum_{i=0}^{\infty}c^{4i}\varphi(2^iu,0)$ is converges.

Now define $$g(u)=\lim_{n\to\infty}2^{-4n}f(2^nu) \quad \text{for all} \quad u\in U.$$

Now taking limit on both sides of inequality (3.12), we have

$$\left\|f(u)-g(u)\right\|\le c^5\sum_{i=0}^{\infty}c^{4i}\varphi(2^iu,0), \quad \text{for all} \quad u\in U.$$

The proof is similar to the proof of the above Theorem.

Corollary 3.2. Let U be a real vector space and V be a complete β-normed space $(0<\beta\le 1)$ and let $\varphi:U\times U\to R^+$ be a function such that

$$\sum_{i=0}^{\infty}16^{-\beta i}\varphi(2^iu,0) \tag{3.14}$$

Converges and $$\lim_{n\to\infty}16^{-\beta n}\varphi(2^nu,2^nv)=0 \quad \text{for all} \quad u,v\in U. \tag{3.15}$$

Suppose that f satisfies

$$\left\|f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)\right\|\le\varphi(u,v) \tag{3.16}$$

for all $u,v\in U$. Then there exist a unique quartic function $g:U\to V$ which satisfies the following equation

$$f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)=0 \quad \text{for all} \quad u,v\in U. \tag{3.17}$$

and the inequality $$\left\|f(u)-g(u)\right\|\le 32^{-\beta}\sum_{i=0}^{\infty}16^{-\beta i}\varphi(2^iu,0) \quad \text{for all} \quad u\in U. \tag{3.18}$$

Function g is given by $$g(u)=\lim_{n\to\infty}2^{-4n}f(2^nu) \quad \text{for all} \quad u,v\in U. \tag{3.19}$$

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