sp-URYSOHN SPACES

N. Bandyopadhyay
Panskura Banamali College,West Bengal, India,

Abstract: In this paper we introduce semi-pre urysohn space using semi-preopen sets introduced by Andrijevic′ [1].Basic properties of semi-pre Urysohn space along with the interrelationship of this new space with some other spaces have been investigated.
2010 Mathematics subject classification : 54C99

Key words: s.p.o., sp-Urysohn , qspi, α-set.

INTRODUCTION

1986, Andrijević [1] introduced the notion of semi- preopen sets and obtained its various properties. Semi-Urysohn spaces using semi-open sets introduced by Levine [8] were defined by Bhamini [5] and further investigated by Arya and Bhamini in [2], [3] and [4]. Paul et.al. [12] defined pre-Urysohn space utilising pre-open set of Mashhour [9]. In this paper we define semi-pre-Urysohn (briefly sp- Urysohn) space with the aid of semi- preopen (briefly s.p.o.) sets and study some basic properties of this space. Some amount of effort has been expended to investigate the interrelationship between this new space with other existing spaces in literature. In section 2 of this paper some known definitions and results necessary for the presentation of the paper are given.

2. PRELIMINARIES

Throughout the paper (X, τ) or X always denotes a non trivial topological space.The family of all open sets containing x is denoted by $\Sigma(x)$. Interior and closure of a subset A of X is denoted by Int(A) and Cl(A) respectively.

Definition 2.1. $A \subset X$ is called a

i) pre-open set [9] (briefly p.o. set) iff $A \subset Int (Cl (A)$;

ii) α -set [10] iff $A \subset Int (Cl (Int(A)))$;

iii) semi-preopen set[1] (briefly s.p.o. set) iff $A \subset Cl (Int (Cl (A)))$.

The family of all p.o.(resp . α, s.p.o.) sets is denoted by PO(X) (resp. $\alpha(X)$, SPO(X)). For each $x \in X$, the family of all p.o.(resp. s.p.o.) sets containing x is denoted by PO(X, x) (resp.SPO (X,x)).

Definition 2.2. The complement of a s.p.o. set is called semi-preclosed[1]. Equivalently a set F is semi-preclosed [1] iff $Int (Cl (Int (A))) \subset F$. The family of all semi-preclosed sets is denoted by SPF (X).

Definition 2.3. The semi-preclosure[1](resp. pre-closure [6]) of $A \subset X$ is denoted by spcl (A) (resp.pcl(A)(and is defined by spcl (A) = $\cap$\{B : B is semi-preclosed and $B \supset A$\}(resp.pcl (A) =$\cap$\{B : B is preclosed and $B \supset A$).

Definition 2.4. A topological space X is said to be sp- T_2 [7] iff for every pair of points x, $y \in X$ such that $x \neq y$, there exist $U \in SPO(X,x)$ and $V \in SPO(X,y)$ such that $U \cap V = \phi$.

Definition 2.5 A function $f : X \rightarrow Y$ is said to be quasi preirresolute [11] briefly qpi iff for each $x \in X$ and for each $V \in PO (Y, f (x))$ there exists a $U \in PO (X, x)$ such that $f [U] \subset pcl_Y (V)$.

Definition 2.6. X is called sp-regular [12] if for each closed set F of X and each $x \notin F$ there exist U, $V \in SPO (X)$ such that $F \subset U$, $x \in V$.

Definition 2.7. X is called pre-Urysohn [12] iff for every pair of points $x, y \in X$ such that $x \neq y$ there exist $U \in PO (X, x)$, $V \in PO (X, y)$ such that $cl (U) \cap pcl (V) = \phi$.

Theorem 2.1 [12]. A topological space X is sp-regular iff for each $x \in X$ and $U \in \Sigma (x)$ there exists $V \in SPO (X)$ such that $x \in V \subset spcl (V) \subset U$.

Lemma 2.1 [7]. If $A \in SPO (X)$, $B \in \alpha (X)$, then $A \cap B \in SPO (X)$.

Lemma 2.2 [7]. Let $A \subset Y \subset X$ and $Y \in PO (X)$, then $A \in SPO (X)$ iff $A \in SPO (Y)$.

sp-Urysohn space and its basic properties.

We start with the following definitions,

Definition 3.1. A space X is called semi-pre-Urysohn (briefly sp-Urysohn) space if for every pair of distinct points $x, y \in X$ there exist $U \in SPO (X, x)$,
$V \in SPO (X, y)$ such that $spcl (U) \cap spcl (V) = \phi$.

Remark 3.1. A Urysohn space is sp-Urysohn. But the converse does not hold as is clear from the following example.

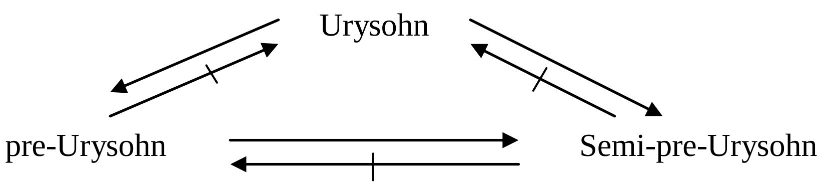
Example 3.1. Let X = \{a, b, c, d\} be equipped with the topology
$\tau = \{\phi, X, \{c, d\}\}$. Then (X, τ) is sp-Urysohn but not Urysohn.

Remark 3.2. A pre-Urysohn space is sp-Urysohn but not conversely as shown by

Example 3.2. Let X = \{a, b, c, d\} be endowed with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is sp-Urysohn but not pre-Urysohn.

R. Paul and P. Bhattacharyya [12] showed that every Urysohn space is pre-Urysohn but the converse is not, in general, true.

The above discussion can be summarized in the following diagram.



Lemma 3.1. For A, B ⊂ X the following hold :

(i) spcl (ϕ) = ϕ, spcl (X) = X;
(ii) A ⊂ spcl (A);
(iii) spcl (A) ⊂ Cl (A) but Cl (A) ⊄ spcl (A);
(iv) A ⊂ B ⇒ spcl (A) ⊂ spcl (B);
(v) spcl (A) ∪ spcl (B) ⊂ spcl (A ∪ B);
(vi) spcl (A ∩ B) ⊂ spcl (A) ∩ spcl (B);
(vii) A is semi preclosed iff spcl (A) = A;
(viii) scpl (A) is the smallest semi-preclosed set containing A;
(ix) x ∈ spcl (A) iff A ∩ U ≠ ϕ, ∀ U ∈ SPO (X,x);
A ∈ SPO (X) iff spint (A) = A and Int (A) ⊂ spint (A).

Proofs. Proofs of (i) → (viii) and that of (x) involve standard arguments as applied in the classical cases and may be left out. For (ix) let us supose x ∈ spcl (A). If possible let U ∈ SPO (X, x) and U ∩ A = ϕ.This gives A ⊂ X – U ∈ SPF (X). Consequently spcl (A) ⊂ X – U and x ∉ spcl (A), contradicting the assumption. Next assume A ∩ G ≠ ϕ, ∀ G ∈ SPO (X,x).If possible, suppose x ∉ spcl (A). Then X – spcl (A) ∈ SPO (X, x) and A ∩ (X – spcl (A)) = ϕ. This is again a contradiction to the assumption. Hence x ∈ spcl (A).

Definition 3.2. A function f : X → Y is termed sp-open if f [U] ∈ SPO (Y) for every U ∈ SPO (X).

Definition 3.3. A function f : X → Y is said to be quasi sp-irresolute (briefly qspi) if for every x ∈ X and every V ∈ SPO (Y, f (x)) there exists a

U ∈ SPO (X, x) such that f [U] ⊂ $spcl_Y$ (V).

Remark 3.3. A qspi map need not be qpi [11] as the following example shows.

Example 3.3. Let X be the space of Example 3.2 and Y = {a, b, c, d} be endowed with σ = {ϕ, {a, b}, Y}. Then the identity map i : X → Y is qspi but not qpi.

Lemma 3.2. Let f : X → Y be a bijective sp-open map. Then

f [F] ∈ SPF (Y) for any F ∈ SPF (X).

Proof. The proof follows directly from the Definition 3.2 and therefore is left out.

Theorem 3.1. A sp-Urysohn space is sp-T_2.

Proof. Let x, y ∈ X, x ≠ y. The sp-Urysohnness of X induces the existence of U ∈ SPO (X, x), V ∈ SPO (X, y) such that spcl (U) ∩ spcl (V) = ϕ.

This gives U ∩ V = ϕ. Hence X is sp-T_2.

Subspaces and transformations

Lemma 3.3. If B ⊂ Y ⊂ X and Y ∈ α (X) then $spcl_Y$ (B) = $spcl_X$ (B) ∩ Y.

Proof. Let y ∈ $spcl_Y$ (B) so that y ∈ Y. Let V ∈ SPO (X, y). The α-ness of Y and semi-pre-open -ness of V together imply, by Lemma 2.1 that Y ∩ V ∈ SPO (X, y). Since every α-set is a p.o. set, Y ∈ PO (X). Now the conditions Y ∩ V ⊂ Y ⊂ X, Y ∈ PO (X) and Y ∩ V ∈ SPO (X, y) together yield, by Lemma 2.3 that Y ∩ V ∈ SPO (Y, y). Hence by Lemma 3.1, (Y ∩ V)∩B ≠ϕ ⇒V∩B ≠ ϕ⇒ y ∈ $spcl_X$ (B) ⇒ y ∈$spcl_X$ (B)∩Y .Consequently, $spcl_Y$ (B) ⊂ $spcl_X$ (B) ∩ Y. To establish the reverse inclusion, let y ∈ $spcl_X$ (B) ∩ Y ⇒ y ∈ $spcl_X$ (B), y ∈ Y.Take V_0 ∈ SPO (Y, y). Pursuing the same reasoning as above, one obtains V_0 ∈ SPO (X, y).Hence by Lemma 3.1,V_0 ∩ B ≠ ϕ⇒ y ∈ $spcl_Y$ (B).Therefore, $spcl_X$ (B) ∩ Y ⊂ $spcl_Y$ (B).

Remark 3.4. The property of being a sp-Urysohn space is not hereditary. It can be seen from the following example.

Example 3.4. Let X be the space of Example 3.1. Then X is sp-Urysohn but the subspace {a, c} of X is not sp-Urysohn.

However the following result holds.

Theorem 3.2. Every α-subspace of a sp-Urysohn space (X, τ) is sp-Urysohn.

Proof. Let Y ⊂ X and Y ∈α (X). Also let y_1, y_2 ∈ Y (y_1 ≠y_2). Clearly y_1, y_2 ∈ X. Sp-Urysohnness of X implies that there exist U ∈ SPO (X, y_1), V ∈ SPO (X, y_2) such that $spcl_X$ (U) ∩ $spcl_X$ (V) = ϕ. Since Y ∈α (X), by lemma 2.3, U ∩ Y ∈ SPO (X, y_1) and V ∩ Y ∈ SPO (X, y_2). Therefore by lemma 3.3 one obtains $spcl_Y$ (U ∩ Y) ∩ $spcl_Y$ (V ∩ Y) = ($spcl_X$ (U ∩ Y) ∩ Y) ∩ ($spcl_X$ (V ∩ Y) ∩ Y) = $spcl_X$ (U ∩ Y) ∩ $spcl_X$ (V ∩ Y) ∩ Y ⊂ $spcl_X$ (U ∩ Y) ∩ $spcl_X$ (V ∩ Y) ⊂ $spcl_X$ (U) ∩ $spcl_X$ (V) = ϕ.Thus $spcl_Y$ (U ∩ Y) ∩ $spcl_Y$ (V ∩ Y) = ϕ. Hence Y is sp-Urysohn.

sp-Urysohn spaces remain invariant under certain bijective mapping as illustrated in the next theorem.

Theorem 3.3. Let f : X → Y be a bijective sp-open map and X is sp-Urysohn. Then Y is sp-Urysohn.

Proof. Let y, z ∈ Y (y ≠ z). The bijectivity of f implies that f^{-1}(y), f^{-1}(z) ∈ X and f^{-1}(y) ≠ f^{-1}(z).Since X is sp-Urysohn there exist U ∈ SPO (X, f^{-1} (y)), V ∈ SPO (X, f^{-1} (z)) such that $spcl_X$ (U) ∩ $spcl_X$ (V) = ϕ.Clearly $spcl_X$ (U), $spcl_X$ (V) ∈ SPF (X).This together with the bijectivity and sp-openness of f imply, by Lemma 3 .2, that f [$spcl_X$ (U)], f [$spcl_X$ (V)] ∈ SPF (X).Obviously f [U] ⊂ f [$spcl_X$ (U)]and f [V] ⊂ f [$spcl_X$ (V)].Since semi-preclosure respects inclusion it follows that $spcl_Y$ (f [U]) ⊂ $spcl_Y$ (f [$spcl_X$ (U)]) = f [$spcl_X$ (U)]and $spcl_Y$ (f [V]) ⊂ $spcl_Y$ (f [$spcl_X$ (V)]) = f [$spcl_X$ (V)] . Hence $spcl_Y$ (f [U]) ∩ $spcl_Y$ (f [V]) ⊂ f [$spcl_X$ (U)] ∩ f [$spcl_X$ (V)].The injectivity of f now assures that $spcl_Y$ (f [U]) ∩ $spcl_Y$ (f [V]) ⊂ f [$spcl_X$ (U) ∩ $spcl_X$ (V)] = f [ϕ] = ϕ. Finally sp-openness of f guarantees that f [U] ∈ SPO (Y, y), f [V] ∈ SPO (Y, z). Hence Y is sp-Urysohn.

Theorem 3.4. If Y is sp-Urysohn and f : X → Y is qspi injection, then X is sp-T_2.

Proof. Let x_1, x_2 (x_1 ≠ x_2) ∈ X.The injectivity of f implies that f (x_1) ≠f (x_2). Since Y is sp-Urysohn there exist H ∈ SPO (Y, f (x_1)), W ∈ SPO (Y, f (x_2)) such that $spcl_Y$ (H) ∩ $spcl_Y$ (W) = ϕ. This gives f^{-1} [$spcl_Y$ (H)] ∩ f^{-1} [$spcl_Y$ (W)] = ϕ.The qspi of f implies the existence of U ∈ SPO (X, x_1), V ∈ SPO (X x_2)such that f [U] ⊂ $spcl_Y$ (H) and f [V] ⊂ $spcl_Y$ (W).Therefore, from the forgoing U_1 ∩ U_2 ⊂ f^{-1} [$spcl_Y$ (H)] ∩ f^{-1} [$spcl_Y$ (W)] = ϕ.Hence X is sp-T_2.

Relation between *sp*-Urysohn and *sp*-regular spaces

Remark 3.5. A sp-regular space is not in general T_2. This is clear from **Example 3.2.** X is the only open set containing c and d and hence X is not T_2. Here SPO (X) = {ϕ, {a}, {b}, {a, b}, {a, c} {a, d} {b, c}, (b, d}, {a, b, c}, {a, c, d}, {b, c, d}, X}. Take a ∉ {b, c, d} which is a closed set. Thus {a}, {b, c, d} are two disjoint s.p.o sets which separate {a} and {b, c, d}. This conclusion is valid for all other points and this leads to the sp-regularity of X.

On the otherhand, the following holds :

Theorem 3.5. Let X be a sp-regular T_2 space. Then X is sp-Urysohn.

Proof. Let $x_1, x_2 \in X$, $(x_1 \neq x_2)$. T_2-ness of X implies that there exist $U \in \Sigma(x_1)$, $V \in \Sigma(x_2)$ such that $U \cap V = \phi$. $\Rightarrow Cl(U) \cap V = \phi$. Now $U \in \Sigma(x_1) \Rightarrow U \in SPO(X, x_1)$.Since X is sp-regular, by Theorem 0.2, there exists $H \in SPO(X, x_2)$ such that $x \in H \subset spcl(H) \subset V$.So from the foregoing $Cl(U) \cap spcl(H) = \phi$. Again by the containment relation between closure and semi-preclosure one gets $spcl(U) \cap spcl(H) = \phi$,where $U \in SPO(X, x_1)$, $H \in SPO(X, x_2)$. Hence X is sp-Urysohn.

REFERENCES

1. D. Andrijević, Semipreopen Sets, Mat. Vensik **38** (1986), 24 – 32.
2. S.P. Arya and M.P.Bhamini, P-Closed spaces,Indian J.Pure Appl. Math.,15(1984),89-98.
3. S.P. Arya and M.P.Bhamini, Some Generalisation of pairwise Urysohn Spaces, Indian J.Pure Appl. Math.,18(1987),1088-1093.
4. S.P. Arya and M.P.Bhamini, Semi- topological properties, Internat J. Math. Sci,15(1992), 267-272.
5. M.P.Bhamini ,The role of Semi-open sets in Topology, Ph.D Thesis, Univ of Delhi,1983.
6. N.El. Deeb, I.A.Hasanein, A.S.Mashhour ,On p-regular spaces, Bull.Math. de la Soc. Sci. Math. de la Roumanie Tome 27 (75), nr. 4, (1983), 311 – 315.
7. P. K.Ghosh, Ph.D. Thesis, University of Kalyani, Kalyani, West Bengal, 2005.
8. N, Levine, Semi-open sets and semi-continuity in topological spaces, Amer.Math. Monthly 70(1963), 36-41.
9. A.S. Mashhour, M.E.Abd. El-Monsef,S.N.El-Deeb, On precontinuous and weak precontinuous mappings, Proc.Math. Soc. Egypt 53 (1982).47-53.
10. O. Njåstad, On some classes of nearly open sets, Pacific. J. Math. 15 (3) (1965), 961 – 970.
11. M.Pal and P.Bhattacharyya, Faint precontinuous function, Soochow J. Math. 21 (3), (1995), 273 – 289.
12. R.Paul. and P.Bhattacharyya, On Pre-Urysohn spaces, Math. Soc. second series 22(1999), 23-24.