Matrix method approach for the temperature distribution and heat flow along a conducting bar connected between two heat sources

[1]Rohit Gupta, [2]Rahul Gupta
Lecturer Physics, Department of Applied Sciences.
Yogananda College of Engineering and Technology (YCET), Jammu (J&K).

Abstract: Heat transfer is a subject of widespread interest to the engineering students and technicians engaged in the design, construction, and operation of equipment required to exchange heat in scientific and industrial technology. In many situations of practical importance, heat is generated at a uniform rate itself within the conducting medium and is lost from its surface to the surroundings. The heat generated has to be controlled, otherwise, the rise in temperature resulting from heat produced within the conducting medium results in the failure of the medium. The distribution of temperature within the uniform conducting bar and the heat that removed by the uniform conducting bar from the heat source maintained at a higher temperature and lost it to the surroundings assume significant importance in the design and construction of thermal systems. In this paper, we will discuss the distribution of temperature and the heat flow along the length of a uniform conducting bar connected between two heat sources maintained at different temperatures by solving the differential equation describing the distribution of temperature along uniform conducting bar via Laplace transform method.

Keywords: Flow of heat, temperature distribution, uniform conducting bar.

Introduction:

Heat transfer is a subject of widespread interest to the engineering students and technicians engaged in the design, construction and operation of equipment required to exchange heat in scientific and industrial technology. Heat is one of the forms of energy which transfers by virtue of temperature gradient from a region at a higher temperature to another region at a lower temperature i.e. it flows in the direction of decreasing temperature, with a negative temperature gradient. There are three modes which transfer heat from one region of the medium to another namely- conduction, convection and radiation. The first two modes of heat transfer are dominant in many practical fields while the radiation mode of heat transfer is significant at high temperatures. In conduction mode, heat energy transfers from a region of the medium at a higher temperature to another region at a lower temperature without any macroscopic motion in the medium. Fourier's law is the basic law of conduction of heat and is expressed in differential form as $H = -ka\frac{dT}{dy}$, where k is the thermal conductivity of the material of the uniform conducting bar, a is the area of the cross-section of the conducting medium, H is the rate of conduction of heat in the direction of flow of heat and $\frac{dT}{dy}$ is the temperature gradient. The negative sign indicates that the heat is flowing in a direction of decrease of temperature. In convection mode, heat energy transfers from one region of medium to another with the macroscopic motion in the medium. In many situations of practical importance, heat is generated at a uniform rate itself within the conducting medium. For example, electrical energy is converted into thermal energy in the current carrying conductor by resistance heating in the conductor, energy is liberated due to exothermic reactions occurring within the medium and so on. The rate of heat generated has to be controlled, otherwise, the rise in temperature resulting from it lead to the failure of the conducting medium. The distribution of temperature within the uniform conducting bar and the heat loss by it to its surroundings assume significant importance in the design and construction of thermal systems. The temperature distribution and the heat flow along the uniform conducting bar connected between two heat sources at different temperatures can be derived by solving the differential equation representing the distribution of temperature along the uniform conducting bar via the Matrix method.

Eigen values and Eigen vectors:

If B is any square matrix of order n with elements b_{ij}, we can obtain a column matrix Y and a constant λ such that BY = λY or BY - λI Y = 0 or $|B - \lambda I|$Y = 0

This matrix equation represents n homogeneous linear equations:

$$(b_{11} - \lambda)\, y_1 + b_{12}\, y_2 + b_{13}\, y_3 + \cdots + b_{1n}\, y_n = 0$$

$$b_{21}\, y_1 + (b_{22} - \lambda)\, y_2 + b_{23}\, y_3 + \cdots + b_{2n}\, y_n = 0$$

……………………………………………

$b_{n1}\, y_1 + b_{n2}\, y_2 + b_{n3}\, y_3 + \cdots + (b_{nn} - \lambda)\, y_n = 0$, which will have a non – trivial solution only if the determinant of the coefficients vanishes i.e.

$$\begin{vmatrix} (b_{11} - \lambda) & b_{12} \ldots\ldots & \ldots .\, b_{1n} \\ b_{21} & (b_{22} - \lambda) \ldots & \ldots .\, b_{2n} \\ \ldots\ldots\ldots . & \ldots\ldots\ldots . & \ldots\ldots\ldots \\ b_{n1} & b_{n2} \ldots\ldots & \ldots .\, (b_{nn} - \lambda) \end{vmatrix} = 0$$

When we expand the determinant we will get n[th] degree equation in λ, which is known as the characteristic equation of the matrix B. The roots of the characteristic equation of matrix B i.e. $\lambda_i\ (where\ i = 1, 2, 3, \ldots\ldots n)$ are called Eigen values or latent roots. Corresponding to each Eigen value, the characteristic equation of matrix B will have a non-zero solution Y = $\begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix}$ (a column matrix), which is known as Eigen vector.

Formulation:

Governing differential equation:

To derive the differential equation describing the distribution of temperature along the uniform conducting bar, we consider a conducting bar having length L, uniform area of cross-section 'a' and perimeter 'P' connected between two heat sources R_1 and R_2 at points y = 0 and y = L. The heat sources are maintained at fixed temperatures T_1 and T_2 respectively. If the temperature of the surroundings of the uniform conducting bar is denoted by T_s and is kept constant, then the convective heat will flow from the bar to the surroundings which lead to a heat loss from the uniform conducting bar to its surroundings.

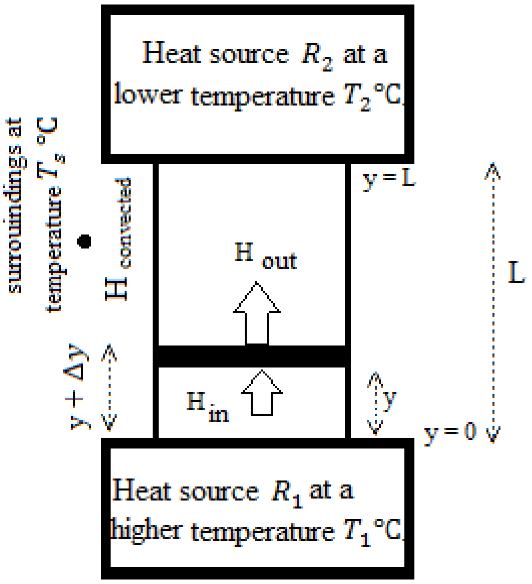

Figure 1: Conduction of heat through a uniform rod with heat loss by convection.

Let us consider an infinitesimal section of thickness Δy of the uniform conducting bar, located at a distance of y from the source R_1 at higher temperature as shown in figure 1.

Heat conducted into the element at plane y is given by

$$(H_y)_{in} = -\text{k a}\,[D_y T(y)]_{\,y} \quad \ldots\ldots\ldots (1)$$

$$D_y \equiv \frac{d}{dy}$$

Where T(y) is the temperature of the uniform conducting bar and is a function of variable y. It is assumed to be constant for the infinitesimal section of the uniform conducting bar.

Heat conducted out of the element at plane (y +Δy) is given by

$$(H_{y+\Delta y})_{out} = -\text{k a}\,[D_y T(y)]_{\,y+\Delta y}$$

Or

$$(H_{y+\Delta y})_{out} = -kaD_y\{T(y) + [D_y T(y)]\,\Delta y\} \quad \ldots . (2)$$

Heat convected of the element of length Δy between the planes at y and y +Δy is given by

$$H_{\text{convected}} = \sigma\ \text{P}\ \Delta y[T(y) - T_s] \ \ldots . (3)$$

Where k is the thermal conductivity of the material of uniform conducting bar and σ is the coefficient of heat transfer by convection and P is the perimeter of the uniform conducting bar.

Making use of steady state heat balance, we can write

$$(H_y)_{in} = (H_{y+\Delta y})_{out} + H_{\text{convected}} \ldots\ldots . (4)$$

Using equations (1), (2) and (3) in equation (4), we get

$$-ka[D_y T(y)] = -\text{k a}D_y\{T(y) + [D_y T(y)]\,\Delta y\} + \sigma\ \text{P}\ \Delta y[T(y) - T_s]$$

Or $-ka[D_yT(y)] =$
$-$ k a$[D_yT(y)]-\ ka[D_y^2T(y)]\,\Delta y + \sigma$ P $\Delta y[T(y) - T_s]$
Or
$-ka[D_y^2T(y)]\,\Delta y + \sigma$ P $\Delta y[T(y) - T_s] = 0$
Upon simplifying the above equation, we get
$D_y^2T(y) - \frac{\sigma P}{ka}[T(y) - T_s] = 0$ ……… (5)
To simplify the equation (5), let us substitute
$(\frac{\sigma P}{ka})^{\frac{1}{2}} = \beta$……. (6)
And define $T(y) - T_s = \tau(y)$ …. (7)
Where $\tau(y)$ is called excess temperature at length y of the bar.
Then $D_y\,[T(y) - T_s] = D_y\tau(y)$.
As T_s is constant, therefore, we can write $D_y^2T(y)$=$D_y^2\tau(y)$,
Therefore, equation (5) can be rewritten as
$D_y^2\tau(y) - \beta^2\,\tau(y) = 0$ ……….. (8)
Equations (5) and (8) are the general form of energy equations for one-dimensional heat loss from the uniform conducting bar.
In equation (6), β is constant provided that σ is constant over the entire surface of uniform conducting bar and k is constant within the range of temperature considered.

Solution of the differential equation:

To solve equation (8) by matrix method, we need to write the necessary boundary conditions as given below:
At y = 0, $T = T_1$ and at y = L, $T = T_2$
In terms of excess temperature (τ), we can write, at y = 0, $T - T_s = T_1 - T_s$ or $\tau = \tau_1$ and at y = L, $T - T_s = T_2 - T_s$ or $\tau = \tau_2$
Let us substitute
$\tau(y) = \theta_1(y)$
And $D_y\theta_1(y) = \theta_2(y)$... (9)
We can rewrite equation (8) as
$D_y\theta_2(y) - \beta^2\theta_1(y) = 0$
Or $D_y\theta_2(y) = \beta^2\theta_1(y)$ … …. (10)
Differential equations (9) and (10) can be written in single matrix form as

$$D_y\begin{bmatrix}\theta_1(y)\\ \theta_2(y)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \beta^2 & 0\end{bmatrix}\begin{bmatrix}\theta_1(y)\\ \theta_2(y)\end{bmatrix}$$

The characteristic equation of $\begin{bmatrix}0 & 1\\ \beta^2 & 0\end{bmatrix}$ is

$$\begin{vmatrix}0-\lambda & 1\\ \beta^2 & 0-\lambda\end{vmatrix} = 0$$

Solving the determinant, we get
$\lambda^2 - \beta^2 = 0$
Or $\lambda = \pm\beta$

Now the Eigen vector for $\lambda = \beta$ is given by

$$\begin{bmatrix}0-\beta & 1\\ \beta^2 & 0-\beta\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Applying elementary transformation $R_2 \to R_2 + \beta R_1$, we can write

$$\begin{bmatrix}-\beta & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

This results
$-\beta y_1 + y_2 = 0$
Or $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1\\ \beta\end{bmatrix}$
And the Eigen vector for $\lambda = -\beta$ is given by

$$\begin{bmatrix}0+\beta & 1\\ \beta^2 & 0+\beta\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Applying elementary transformation $R_2 \to R_2 - \beta R_1$, we can write

$$\begin{bmatrix}\beta & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

This results
$\beta y_1 + y_2 = 0$
Or $\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1\\ -\beta\end{bmatrix}$

The matrix of Eigen vectors is $\begin{bmatrix}1 & 1\\ \beta & -\beta\end{bmatrix}$.

Let P = $\begin{bmatrix}1 & 1\\ \beta & -\beta\end{bmatrix}$, then the inverse matrix of P is given by $P^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2\beta}\\ \frac{1}{2} & \frac{-1}{2\beta}\end{bmatrix}$.

Now we have to find $Pe^{\lambda y}P^{-1}$.

$$Pe^{\lambda y}P^{-1} = \begin{bmatrix}1 & 1\\ \beta & -\beta\end{bmatrix}\begin{bmatrix}e^{\beta y} & 0\\ 0 & e^{-\beta y}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2\beta}\\ \frac{1}{2} & \frac{-1}{2\beta}\end{bmatrix}$$

$$= \begin{bmatrix}e^{\beta y} & e^{-\beta y}\\ \beta e^{\beta y} & -\beta e^{-\beta y}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2\beta}\\ \frac{1}{2} & \frac{-1}{2\beta}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}(e^{\beta y} + e^{-\beta y}) & \frac{1}{2\beta}(e^{\beta y} - e^{-\beta y})\\ \frac{\beta}{2}(e^{\beta y} - e^{-\beta y}) & \frac{1}{2}(e^{\beta y} + e^{-\beta y})\end{bmatrix}$$

$$= \begin{bmatrix}cosh\beta y & \frac{1}{\beta}sinh\beta y\\ \beta sinh\beta y & cosh\beta y\end{bmatrix}$$

Applying initial condition i.e. $\theta_1(0) = \tau(0) = \tau_1$ and since $\theta_2(0) = D_y\tau(0)$ is a constant, therefore substituting $\theta_2(0) = D_y\tau(0)$ = C (a constant), we can write

$$\begin{bmatrix}\theta_1(y)\\ \theta_2(y)\end{bmatrix} = \begin{bmatrix}cosh\beta y & \frac{1}{\beta}sinh\beta y\\ \beta sinh\beta y & cosh\beta y\end{bmatrix}\begin{bmatrix}\tau_1\\ C\end{bmatrix}$$

$$\text{Or}\ \begin{bmatrix}\theta_1(y)\\ \theta_2(y)\end{bmatrix} = \begin{bmatrix}\tau_1 \cos h\beta y + \frac{C}{\beta}sinh\beta y\\ \tau_1\beta sinh\beta y + Ccosh\beta y\end{bmatrix}$$

This equation results

$\theta_1(y) = \tau_1 \cos h\beta y + \frac{C}{\beta} sinh\beta y$

Or $\tau(y) = \tau_1 \cos h\beta y + \frac{C}{\beta} sinh\beta y$….. (11)

And $\theta_2(y) = \tau_1\beta sinh\beta y + Ccosh\beta y$

Or $D_y\tau(y) = \tau_1\beta sinh\beta y + Ccosh\beta y$ ………… (12)

Determination of the constant C:

To find the value of constant C, applying boundary condition: at y = L, $\tau(y) = \tau_2$, equation (11) provides,

$\tau_2 = \frac{C}{\beta}\sin h\beta L + \tau_1 \cos h\beta L$

Upon rearranging and simplification of above equation, we get

$C = \frac{\beta[\tau_2 - \tau_1 \cos h\beta L]}{\sin h\beta L}$…… (13)

Substitute the value of C from equation (13) in equation (12), we get

$\tau(y) = \frac{\tau_2 - \tau_1 \cos h\beta L}{\sin h\beta L}\sin h\beta y + \tau_1 \cos h\beta y$

Or $\tau(y) = \frac{\tau_2 \sin h\beta y - \tau_1 \cos h\beta L \sin h\beta y + \tau_1 \cos h\beta y \sin h\beta L}{\sin h\beta L}$

Or $\tau(y) = \frac{\tau_2 \sin h\beta y + \tau_1 \sin h\beta(L-y)}{\sin h\beta L}$

Or $\tau(y) = \frac{\tau_1 \sin h\beta(L-y) + \tau_2 \sin h\beta y}{\sin h\beta L}$…. (14a)

Using equation (7), we can write

$T(y) - T_s = \frac{(T_1 - T_s)\sin h\beta(L-y) + (T_2 - T_s)\sin h\beta y}{\sin h\beta L}$

Or $T(y) = T_s + \frac{(T_1 - T_s)\sin h\beta(L-y) + (T_2 - T_s)\sin h\beta y}{\sin h\beta L}$

Put the value of β from equation (6), we get

Or $T(y) = T_s + \frac{(T_1 - T_s)\sin h\sqrt{\frac{\sigma P}{ka}}(L-y) + (T_2 - T_s)\sin h\sqrt{\frac{\sigma P}{ka}}y}{\sin h\sqrt{\frac{\sigma P}{ka}}L}$ …….. (14b)

Equation (14) provides the distribution of temperature along the uniform conducting bar connected between two heat sources at different fixed temperatures T_1 and $T_2 (T_1 > T_2)$ and reveals that the temperature of the uniform conducting bar decreases along its length with the increase in distance from the heat source R_1 at a higher temperature T_1.

The most important parameter is the total amount of heat that can be removed by the uniform conducting bar from the heat source at a higher temperature T_1 and lost by it to the surroundings. The total heat (H) emitted from the surface of the uniform conducting bar to its surroundings can be calculated by integrating the expression for heat convected from the surface of an infinitesimal section of the uniform conducting bar to its surroundings i.e.

H = $\int_0^L \sigma$ P $dy(T(y) - T_s)$

Or H = $\int_0^L \sigma$ P $\tau(y)dy$

Or H = $\int_0^L \sigma$ P $\frac{\tau_1 \sin h\beta(L-y) + \tau_2 \sin h\beta y}{\sin h\beta L}dy$

On solving the integration and applying the limits, we get

H = σ P $\frac{(\tau_1 + \tau_2)(\cos h\beta L - 1)}{\beta \sin h\beta L}$………. (15)

Put the value of β from equation (6) in equation (15), we get

H = σ P $\frac{(\tau_1 + \tau_2)[\cos h\sqrt{\frac{\sigma P}{ka}}L - 1]}{\sqrt{\frac{\sigma P}{ka}}\sin h\sqrt{\frac{\sigma P}{ka}}L}$

Or H = $\sqrt{\sigma Pka}\frac{(\tau_1 + \tau_2)(\cos h\sqrt{\frac{\sigma P}{ka}}L - 1)}{\sin h\sqrt{\frac{\sigma P}{ka}}L}$

Or H = $\sqrt{\sigma Pka}\frac{(T_1 + T_2 - 2T_s)[\cos h\sqrt{\frac{\sigma P}{ka}}L - 1]}{\sin h\sqrt{\frac{\sigma P}{ka}}L}$ ….. (16)

This equation (16) provides the total amount of heat emitted from the surface of the uniform conducting bar to its surroundings and reveals that the heat flow rate can be increased by increasing the surface of the bar across which the convection of heat occurs.

Conclusion: In this paper, we have derived the distribution of temperature along the length of the uniform conducting bar connected between two heat sources maintained at different temperatures and the total amount of heat emitted from the surface of uniform conducting bar to its surroundings by solving the differential equation describing the distribution of temperature along the conducting bar via matrix method. We concluded that the temperature of the uniform conducting bar decreases with increase in distance from the heat source at a higher temperature. We also concluded that the total amount of heat emitted from the bar to its surroundings can be increased by increasing the surface of the bar across which the convection of heat occurs.

References:

[1] Advanced Engineering Mathematics by Erwin Kreysig 10th edition, 2014

[2]Higher Engineering Mathematics by Dr.B.S.Grewal 43rd edition 2015.
[3] Advanced engineering mathematics by H.K. Dass. Edition: Reprint, 2014.
[4] An Introduction to Laplace transforms and Fourier series by Dyke and Phil.
[5] Heat and mass transfer by Dr. D.S. Kumar, 7th revised edition.
[6] Heat and mass transfer by P.K. Nag. 3rd edition 2011.
[7] Heat transfer, principles and applications by Binay K. Dutta.
[8] Convection heat transfer by Adrian Bejan. 4th edition 2013.
[9] Heat transfer by J.P. Holman, 10th edition.
[10] Numerical methods by Dr. B.S. Grewal. Edition 2012.
[11] A book of Engineering Thermodynamics by R.K. Rajput.