ON THE CONSERVATION LAWS OF DEFORMED COUPLED Kdv TYPE NONLINEAR SCHRODINGER EQUATION

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Abstract : In this article, we consider deformed coupled Kdv type nonlinear Schrödinger equation(cKdv type NLSE) and derive its conserved vector components in two ways. We also present Lie point symmetries of deformed cKdv type NLSE. We show that the deformed cKdv type NLSE possesses infinitely many conservation laws which indicates the complete integrability of it.

***IndexTerms* - deformed kdv type nonlinear Schrödinger equation, Lie point symmetries, conservation laws.**

I. INTRODUCTION

It is well known that Conservation laws appear in many areas of the applied sciences, such as quantum physics, electromagnetism, plasma physics, physical chemistry, nonlinear optics[4,5,8,10,11]. Conservation laws play an important role in the resolution of problems in which certain physical properties do not change in the course of time and it plays an important role in the analysis of basic properties of solutions. They are used for analysis, in particular, existence, uniqueness, and stability analysis and construction of numerical schemes. Moreover, conservation laws are used in obtaining the new nonlocal symmetries, nonlocal conservation laws. It was proven that, the existence of a large number of conservation laws of a partial differential equation(PDE) is a strong indication of its integrability. There are various methods to compute conservation laws of nonlinear partial differential equations(NLPDEs): The direct classical method, Noethers approach, the multiplier method, double reduction method, etc,[4,5,8,10-14]. A basic approach depends on the link between symmetries and conservation laws as stated in Noethers theorem. In this paper we show that deformed coupled Kdv type nonlinear Schrödinger equation possesses infinitely many conservation laws. The deformed coupled Kdv type nonlinear Schrödinger equation along with Lax pair[9] is given by

$$u_t + u_{xxx} + 6uvu_x - g = 0 \qquad (1a)$$
$$v_t + v_{xxx} + 6uvv_x - h = 0 \qquad (1b)$$
$$g_x + 2fu = 0 \qquad (1c)$$
$$h_x + 2fv = 0 \qquad (1d)$$
$$f_x - uh - vg = 0 \qquad (1e)$$

where u,v,g,h,f are functions of x,t.

$$A = -i\lambda\sigma_3 + A_0, \qquad (2a)$$
$$B = \lambda B_1 + B_0 + \frac{i}{\lambda}B_{-1}, \qquad (2b)$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} r_x q - rq_x & 0 \\ 0 & -r_x q + rq_x \end{pmatrix} + B_2,$$

$$B_2 = \begin{pmatrix} 0 & -q_{xx} - 2q^2 r \\ r_{xx} & 2r^2 q \end{pmatrix},$$
$$B_1 = 2iqr\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_{0x},$$
$$B = \frac{1}{2}\begin{pmatrix} -f & -g \\ -h & f \end{pmatrix}$$

and A,B satisfying the Lax equation

$$A_t - B_x + [A, B] = 0. \tag{3}$$

The outline of the paper is as follows. In Sect. 2, Using Lie symmetry method we obtain point symmetries of deformed coupled Kdv type nonlinear Schrödinger equation (1). Then in Sect. 3. We construct conservation laws for (1) in two ways (i) by direct calculation with the help of Lax pair (ii) using Lie point symmetries. Finally, in Sect. 4 concluding remarks are presented. In appendix, we present few more conserved vector components.

I. LIE POINT SYMMETRIES

At the end of the 19th century, the symmetry study which laid the foundations by Sophus Lie which plays an important role in almost all the scientific fields. The theory of Lie groups for obtaining the group invariant solutions to NLPDEs is widely recognized as one of the most powerful methods. We observe a plenty of books and survey articles about Lie groups method [1-3,6,7,15-17]. Now let us consider a system of partial differential equations as follows

$$\Delta_m\left(x, u^{(n)}\right) = 0, \qquad m = 1,2, \dots, l. \tag{4}$$

where $u = \left(u_1, u_2, \dots, u_q\right), x = \left(x_1, x_2, \dots, x_p\right), u^n$ denotes all the derivatives of u of all orders from 0 to n. The one-parameter Lie group of infinitesimal transformations of the system (4) is given by

$$x_i^* = x_i + \varepsilon\xi_i(x, u) + O(\varepsilon^2); \qquad i = 1,2, \dots, p,$$
$$u_{*\alpha} = u_\alpha + \varepsilon\eta_\alpha(x, u) + O(\varepsilon^2); \qquad \alpha = 1,2, \dots, q, \tag{5}$$

where ε is the group parameter. The Lie algebra of (4) is spanned by vector field

$$X = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \eta_\alpha(x, u)\frac{\partial}{\partial u_\alpha} \tag{6}$$

The n-th order prolongation of X is given by:

$$X^n = \sum_{i=1}^{p} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \eta_\alpha(x, u)\frac{\partial}{\partial u_\alpha} + \sum_{\alpha=1}^{q}\sum_{J} \eta_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_\alpha^J} \tag{7}$$

Where $J = (i_1, \dots, i_k)$, $1 \le i_k \le p, 1 \le k \le n$, and the sum is over all J's of order $0 < J \le n$.
If J=k, the coefficient η_α^J of $\frac{\partial}{\partial u_\alpha^J}$ depend only on k-th and lower order derivatives of u,
and

$$\eta_\alpha^J\left(x, u^{(n)}\right) = D_J(\eta_\alpha - \sum_{i=1}^{p} \xi_i u_\alpha^i) + \sum_{i=1}^{p} \xi_i u_\alpha^{J,i}\,, \tag{8}$$

where $u_\alpha^i = \dfrac{\partial u_\alpha}{\partial x^i}$ and $u_\alpha^{J,i} = \dfrac{\partial u_\alpha^J}{\partial x_i}$.

By considering the third prolongation of the above vector field, under the constraints that the equations at hand be satisfied. The determining system lead to that the deformed coupled kdv type NLSE (1) is invariant under a one parameter (ϵ) continuous point transformations,

$$x^* = x + \epsilon\xi_1 + O(\epsilon^2),$$
$$t^* = t + \epsilon\xi_2 + O(\epsilon^2),$$
$$u^* = u + \epsilon\eta_1 + O(\epsilon^2),$$
$$v^* = v + \epsilon\eta_2 + O(\epsilon^2),$$
$$g^* = g + \epsilon\eta_3 + O(\epsilon^2),$$
$$h^* = h + \epsilon\eta_4 + O(\epsilon^2),$$
$$f^* = f + \epsilon\eta_5 + O(\epsilon^2),$$

where $\xi_1, \xi_1, \eta_1, \eta_2, \eta_3, \eta_4$ and η_5 are functions of x,t,u,v,g,h,f and v,h the conjugate of u,g respectively with the infinitesimal generator

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} + \eta_3\frac{\partial}{\partial g} + \eta_4\frac{\partial}{\partial h} + \eta_5\frac{\partial}{\partial f}$$

where

$$\xi_1 = c_1 x + c_2,$$
$$\xi_2 = 3c_1 t + c_3$$
$$\eta_1 = (-c_1 + c_4)u,$$
$$\eta_2 = -(c_1 + c_4)v$$
$$\eta_3 = (-4c_1 + c_4)g,$$
$$\eta_4 = -(4c_1 + c_4)h,$$
$$\eta_5 = -4c_1 f \qquad (9)$$

where c_1, c_2, c_3 and c_4 are constants, provided any solution of the dependent variables u,v,g,h,f satisfy the system (1.1). Then the infinitesimal generator X becomes

$$X = (3c_1t + c_3)\frac{\partial}{\partial t} + (c_1x + c_2)\frac{\partial}{\partial x} + (-c_1 + c_4)u\frac{\partial}{\partial u} - (c_1 + c_4)v\frac{\partial}{\partial v} + (-4c_1 + c_4)g\frac{\partial}{\partial g} - (4c_1 + c_4)h\frac{\partial}{\partial h} - 4c_1 f\frac{\partial}{\partial f}$$

For the above point transformations the infinitesimal generators are

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = \frac{\partial}{\partial t},$$
$$X_3 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + g\frac{\partial}{\partial g} - h\frac{\partial}{\partial h}$$
$$X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 4g\frac{\partial}{\partial g} - 4h\frac{\partial}{\partial h} - 4f\frac{\partial}{\partial f}.$$

II. Conservation Laws

In this section, we consider deformed coupled Kdv type nonlinear Schrodinger equation (1) and derive infinitely many conservation laws by (i) using direct calculation with the help of Lax pair (ii) using Lie point symmetries.

Now, we consider the Lax pair given in (2) for the deformed Kdv type nonlinear Schrödingerequation (1). Using the compatibility condition one can construct infinitely many conservation laws of deformed Kdv type nonlinear Schrödinger equation (1). From the compatibility condition we can easily see that the following conserved vector components ρ_k and J_k (Conserved densities and conserved fluxes respectively) identically satisfies

$$\frac{\partial \rho_k}{\partial t} = \frac{\partial J_k}{\partial x}$$

The conserved densities along with its conserved fluxes are given by

$\rho_1 = uv,$

$J_1 = -f + 3u^2v^2 + vu_{xx} - u_xv_x + uv_{xx},$

$\rho_2 = uv_x,$

$J_2 = 6u^2vv_x + u_{xx}v_x - gv + uv_{xxx} - u_xv_{xx},$

$\rho_3 = u^2v^2 + uv_x,$

$$J_3 = 4u^3v^3 + 8u^2vv_{xx} + 5u^2v_x^2 + 2uv^2u_{xx} + 2uvv_xu_x - v^2u_x^2 + uv_{xxxx} + u_{xx}v_{xx} - gv_x - u_xv_{xxx},$$

$\rho_4 = u(4uvv_x + u_xv^2v_{xxx}),$

$$J_4 = -uv^2g - gv_{xx} + 24u^3v^2v_x + 6u^2v^3u_x + 10u^2vv_{xxx} + 12uvu_{xx}v_xu_{xx}v_{xxx} + 6uvu_xv_{xx} + 6uu_xv_x^2 - 6u_x^2vv_x - u_xv_{xxxx} + 18u^2v_xv_{xx}uv_{xxxxx} + uv^2u_{xxx},$$

$\rho_5 = u(2u^2v^3 + 6uvv_{xx} + 5v_x^2 + v^2u_{xx} + 6vu_xv_x + v_{xxxx}),$

$$J_5 = 60u^2v^2u_xv_x + v^2u_{xx}^2 + 10uvu_{xxx} + v_{xxxx}u_{xx} - gv_{xxx} - 11v_x^2u_x^2 - v_{xxxxx}u_x + 9u^4v^4 + 19u^2v_{xx}^2 + uv_{xxxxxx} - v^2u_xu_{xxx} - v^2gu_x - 12vu_x^2v_{xx} + 60u^3vv_x^2 - 12u^2vv_{xxxx} + 42u^3v^2v_{xx} + 12u^2v^3u_{xx} + 24uv_x^2u_{xx} + uv^2u_{xxxx} + 28u^2v_xv_{xxx} + 34uv_xu_xv_{xx} + 26uvu_{xx}v_{xx} - 2vu_xv_xu_{xx} - 4uvv_xg + 12uvu_xv_{xxx},$$

$$\rho_6 = u(u^2v^2v_x + 66uv^3u_x + 8uvv_{xxx} + 18uv_xv_{xx} + v^2u_{xxx} + 8vv_xu_{xx} + 12vv_{xx}u_x + 11u_xv_x^2 + v_{xxxxx},$$

$$J_6 = uv_{xxxxxxx} - 6u_x^3v^3 + u_{xx}v_{xxxxx} - gv_{xxxx} + 60u^3v_x^3 - u_xv_{xxxxxx} - 5guv_x^2 - gv^2u_{xx} - 8u_xv_x^2u_{xx} - 52u_x^2v_xv_{xx} - u_xv^2u_{xxxx} - 20u_x^2vv_{xxx} + v^2u_{xx}u_{xxx} + 8vv_xu_{xx} - 52u_x^2v_xv_{xx} - u_xv^2u_{xxxx} - 20u_x^2vv_{xxx} + v^2u_{xx}u_{xxx} + 8vv_xu_{xx}^2 + 29uv_x^2u_{xxx} + 68u^2v_{xx}v_{xxx} + 12u^2v^3u_{xxx} + 52uv_{xx}^2u_x + 36u^3v^4u_x + 96u^4v^3v_x + 64u^3v^2v_{xxx} + 40u^2v_xv_{xxxx} + 14u^2vv_{xxxxx} + uv^2u_{xxxxx} - 10u_xvv_xu_{xxx} - 6guvv_{xx} - 6gvv_xu_x + 72uu_xv_xv_{xxx} + 20uvu_xv_{xxxx} + 234u^2vv_x^2u_x + 252u^3vv_xv_{xx} + 18uv^3u_xu_{xx} + 138u^2v^2u_{xx}v_x + 128uuv_xu_{xx}v_{xx} + 48uvu_{xx}v_{xxx} + 30uvu_{xxx}v_{xx} + 18uv^2u_x^2v_x + 12uvu_{xxxx}v_x + 144u^2v^2u_xv_{xx}$$

$$-8vu_xu_{xx}v_{xx}-2gu^2v^3$$

....

We also present another set conserved vector components of deformed Kdv type nonlinear Schrödinger equation (1) in Appendix A.

Next, we derive conserved vector components of deformed Kdv type nonlinear Schrödinger equation (1) by using Lie point symmetries.

Definition. The Noether operators associated with a Lie point infinitesimal generator X are

$$N_i=\xi_i+W_\alpha\frac{\delta}{\delta u_\alpha^i}+\sum_{s\geq 1}^{\infty}D_{i_1}\dots D_{i_k}\frac{\partial}{\partial u_\alpha^{i_1\dots i_k}},\quad k=1,\dots,n, \tag{10}$$

in which W_α is the Lie characteristic function

$$W_\alpha=\eta_\alpha-\xi_j u_\alpha^j, \tag{11}$$

where $\frac{\delta}{\delta u_\alpha}$ is the Euler operator which is given by,

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u_\alpha^i}+\sum_{s\geq 1}^{\infty}D_{i_1}\dots D_{i_k}\frac{\partial}{\partial u_\alpha^{i_1\dots i_k}},\quad \alpha=1,\dots,q. \tag{12}$$

Theorem. Every Lie point symmetry gives a conservation law for the equation under consideration. The conserved vector components are determined with

$$\rho_i=\xi_iL+W_\alpha\left(\frac{\partial L}{\partial u^i}-D_j\frac{\partial L}{\partial u^{ij}}+D_jD_k\frac{\partial L}{\partial u^{ijk}}-D_jD_kD_m\frac{\partial L}{\partial u^{ijkm}}\right)$$
$$+D_j(W_\alpha)\left(\frac{\partial L}{\partial u^{ij}}-D_k\frac{\partial L}{\partial u^{ijk}}+D_kD_m\frac{\partial L}{\partial u^{ijkm}}\right)$$
$$+D_jD_k(W_\alpha)\left(\frac{\partial L}{\partial u^{ijk}}-D_m\frac{\partial L}{\partial u^{ijkm}}\right)+D_jD_kD_m\frac{\partial L}{\partial u^{ijkm}} \tag{13}$$

Where L is the Lagrangian function ξ_i and η_α are the coefficient functions of the associated infinitesimal generator.

Now, let us consider infinitesimal point generator $X_1=\frac{\partial}{\partial x}$. For this infinitesimal generator, the infinitesimals are

$$\xi_1=1,$$
$$\xi_2=0$$

and we get two Lie characteristic functions given by

$$W_1=-q_x,$$
$$W_2=-r_x.$$

From the Lie characteristic functions W_1 and W_2 we can easily see that the conserved vector components of deformed Kdv type nonlinear Schrödinger equation (1) are given by

$$\rho_1=-u_x$$
$$J_1=g-u_{xxx}-6uvu_x$$

$$\rho_2=-v_x$$
$$J_1=h-v_{xxx}-6uvv_x$$

For the infinitesimal generator,

$$X_3=u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}+g\frac{\partial}{\partial g}-h\frac{\partial}{\partial h}$$

we obtain the null conserved vectors. Similarly, we can obtain the conserved vector components for other infinitesimal generators.

III. CONCLUSION

We also derived three different sets conserved vector components of deformed ckdv type NLSE(1) by (i) using direct calculation with the help of Lax pair (ii) using Lie point symmetries and which showed that deformed ckdv type NLSE (1) possesses infinitymany conservation laws.

IV.APPENDIX A

Here, we present another set of conserved vector components of deformed ckdv type NLSE (1). The conserved vector components are

$\rho_1=vu,$

$J_1=-f+3v^2u^2+uv_{xx}-v_xu_x+vu_{xx},$

$\rho_2=vu_x,$

$J_2=6v^2uu_x+v_{xx}u_x-hu+vu_{xxx}-v_xu_{xx},$

$\rho_3=v^2u^2+vu_x,$

$$J_3=4v^3u^3+8v^2uu_{xx}+5v^2u_x^2+2vu^2v_{xx}+2vuu_xv_x-u^2v_x^2+vu_{xxxx}+v_{xx}u_{xx}$$
$$-gu_x-v_xu_{xxx},$$

$\rho_4 = v(4vuu_x + v_x u^2 u_{xxx}),$

$$J_4 = -vu^2h - hu_{xx} + 24v^3u^2u_x + 6v^2u^3v_x + 10v^2uu_{xxx} + 12vuv_{xx}u_xv_{xx}u_{xxx} +6vuv_xu_{xx} + 6vv_xu_x^2 - 6v_x^2uu_x - v_xu_{xxxx} + 18v^2u_xu_{xx}vu_{xxxxx} + vu^2v_{xxx},$$

$\rho_5 = u(2v^2u^3 + 6vuu_{xx} + 5u_x^2 + u^2v_{xx} + 6uv_xu_x + u_{xxxx}),$

$$J_5 = 60v^2u^2v_xu_x + u^2v_{xx}^2 + 10vuv_{xxx} + u_{xxxx}v_{xx} - hu_{xxx} - 11u_x^2v_x^2 - u_{xxxxx}v_x +9v^4u^4 + 19v^2u_{xx}^2 + vu_{xxxxxx} - u^2v_xv_{xxx} - u^2hv_x - 12uv_x^2u_{xx} + 60v^3uu_x^2 -12v^2uu_{xxxx} + 42v^3u^2u_{xx} + 12v^2u^3v_{xx} + 24vu_x^2v_{xx} + vu^2v_{xxxx} + 28v^2u_xu_{xxx} +34vu_xv_xu_{xx} + 26vuv_{xx}u_{xx} - 2uv_xu_xv_{xx} - 4vuu_xh + 12vuv_xu_{xxx},$$

$$\rho_6 = v(v^2u^2u_x + 66vu^3v_x + 8vuu_{xxx} + 18vu_xu_{xx} + u^2v_{xxx} + 8uu_xv_{xx} + 12uu_{xx}v_x +11v_xu_x^2 + u_{xxxxx},$$

$$J_6 = vu_{xxxxxxx} - 6v_x^3u^3 + v_{xx}u_{xxxxx} - hu_{xxxx} + 60v^3u_x^3 - v_xu_{xxxxxx} - 5hvu_x^2 -hu^2v_{xx} - 8v_xu_x^2v_{xx} - 52v_x^2u_xu_{xx} - v_xu^2v_{xxxx} - 20v_x^2uu_{xxx} + u^2v_{xx}v_{xxx} +8uu_xv_{xx} - 52v_x^2u_xu_{xx} - v_xu^2v_{xxxx} - 20v_x^2uu_{xxx} + u^2v_{xx}v_{xxx} + 8uu_xv_{xx}^2 +29vu_x^2v_{xxx} + 68v^2u_{xx}u_{xxx} + 12v^2u^3v_{xxx} + 52vu_{xx}^2v_x + 36v^3u^4v_x +96v^4u^3u_x + 64v^3u^2u_{xxx} + 40v^2u_xu_{xxxx} + 14v^2uu_{xxxxx} + vu^2v_{xxxxx} -10v_xuu_xv_{xxx} - 6hvuu_{xx} - 6huu_xv_x + 72vv_xu_xu_{xxx} + 20vuv_xu_{xxxx} +234v^2uu_x^2v_x + 252v^3uu_xu_{xx} + 18vu^3v_xv_{xx} + 138v^2u^2v_{xx}u_x + 128vvu_xv_{xx}u_{xx} +48vuv_{xx}u_{xxx} + 30vuv_{xxx}u_{xx} + 18vu^2v_x^2u_x + 12vuv_{xxxx}u_x + 144v^2u^2v_xu_{xx} -8uv_xv_{xx}u_{xx} - 2hv^2u^3.$$

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