DISTANCE IN FUZZY DIGRAPHS

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Abstract: In this Paper, we introduce a concept of Distance in fuzzy digraphs, which is a metric in fuzzy digraphs. Based on this metric the concepts of eccentricity, radius, diameter and centered fuzzy digraphs are studied. Also obtained the some properties of Eccentric nodes, peripheral nodes and central nodes. Two other distance in fuzzy digraphs are the maximum distance md_D (u,v) and the sum distance in fuzzy digraph sd_D (u,v). Several results and problems concerning these metrices are described.

Keywords: Fuzzy digraph, maximum distance, sum distance.

1. INTRODUCTION

In 1965 Lofti.A.Zadeh [13] introduced a mathematical frame work to explain the concept of uncertainty in real life through the partication of a seminal paper. The Theory of fuzzy graph was developed by Rosenfeld (1975) in the year 1975. In a fuzzy digraph $G_D : (V, \sigma_D, \mu_D,), d_D(u, v)$ is a metric on V_D. F.Hasary et al. in [1] propose the concept of the dissimillarity characteristic of Husimi trees. F.Buckley [2] analysed the concept of Distance in graphs. H. Bielax et al. in [4] identified the vertices in graphs. G.Chartrand et. al.in [9] analyzed the concept of the maximum distance in digraphs. S Tian et.al.in [10] analyzed the concept of Appendage numbers of digraph. S. Tian et.al.in [12] proposed the concept of sum distance in digraphs.

In this paper, we review the concept of fuzzy diagraphs and also we present the brief idea of distance in fuzzy diagraph and its classification. Also we discuss some of the concepts involving maximum distance in fuzzy digraph and the sum distance in fuzzy digraph is defined and proved that it is a metric. Based on this metric eccentricity, radius diameter, center in fuzzy digraph are defined in the distance in fuzzy diagraphs.

2. PRELIMINARIES

Definition: 2.1

A fuzzy graph $G = (V, \sigma, \mu)$ where v is the vertices, σ is a fuzzy subset of v and μ is a membership value on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for every $u, v \in V$.

Definition:2.2

A fuzzy diagraph $G_D = (\sigma_D, \mu_D)$ is a pair of functions $\sigma_D: V \to [0,1]$ and $\mu_D: VXV \to [0,1]$ where $\mu_D \leq \sigma_D\ (u) \wedge \sigma_D(v)$ for $u, v \in v$ and μ_D is a set of fuzzy directed edges called the fuzzy arcs.

Definition: 2.3

Let G_D be a strong fuzzy diagraph, The distance $\vec{d}_D(u, v)$from a vertex u to vertex v in G_D is the length of a shortest u-v path.

The distance is not a metric. $\vec{d}_D[\sigma_D(u), \sigma_D(v)] = \max_{u,v\in V} [\mu\ (u, v)]$

Definition:2.4

For vertices $\sigma_D(u)$ and $\sigma_D(v)$ in a strong fuzzy diagraph G_D. The maximum distance between $\sigma_D(u)$ and $\sigma_D(v)$was defined by a fuzzy graph as the length of the shortest path between them.

$m\vec{d}_D[\sigma_D(u), \sigma_D(v)] = \max [\mu_D\ (u, v)] \forall\, u, v \in v$

Definition:2.5

The sum distance $sd_D[\sigma_D(u, \sigma_D(v)]$ between two nodes $\sigma_D(u)$ and $\sigma_D(v)$ in a fuzzy diagraph G_D as the length of the shortest path them. $S\vec{d}_D[\sigma_D(u), \sigma_D(v)] = \min\ [\mu_D\ (u, v)] \forall\, u, v \in V$.

3. DISTANCE IN FUZZY DIGRAPH

Definition: 3.1

Let G_D be a strong fuzzy diagraph. The distance $\vec{d}_D\ (u, v)$ from a vertex u to vertex v in G_D is the length of a shortest path u-v.

$$\vec{d}_D\ [\sigma_D(u), \sigma_D(v)] = \max_{u,v\in v.} [\mu_D\ (u, v)]$$

Definition:3.2

Let $G_D = (V, \sigma_D, \mu_D)$ be a connected fuzzy diagraph and $e(v_D)$ of a vertex v_D in a fuzzy diagraph G_D is the distance v to a vertex farthest from v.

$$e(v_D) = \max_{u,v \in v(G_D)} \left(\vec{d}_D(u,v)\right)$$

Definition: 3.3

The radius rad G_D of G_D is the minimum eccentricity among the vertices of G_D.

rad $G_D = \min_{v \in v(G_D)} e(v_D)$.

Definition: 3.4

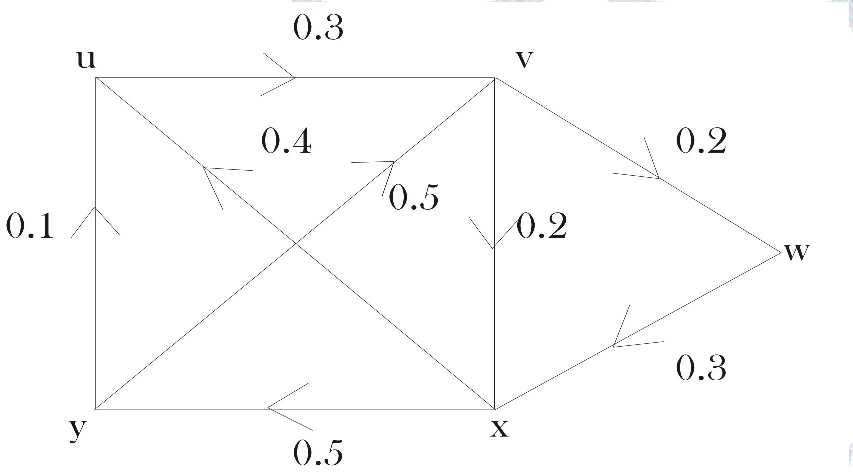
The diameter of a fuzzy diagraph G_D is denoted by diam $G_D = \max_{(v \in v(G_D)} e(v_D)$.

Remark

If $\mu(u,v) = 1 \ \forall (u,v) \in \ \mu_D$ then $\vec{d}_D\ (u,v)$ is the length of the shortest path as in crisp graph.

Definition:3.5

The fuzzy sub diagraph induced by $c(G_D)$denoted by $< c(G_D) = H : (\ v, \tau_D, v_D)$ is called the center of G_D. A connected fuzzy diagraph G is self centered if each node is a central node, ie) $G_D \cong H_D$. A node u is a peripheral node if $e(u) = d(G_D)$.



Example : 3.6

In figure $\vec{d}_D(u,v) = 0.3, \vec{d}_D(v,w) = 0.2, \vec{d}_D(v,x) = 0.2, \vec{d}_D(w,x) = 0.3$,
$\vec{d}_D(y,v) = 0.5, \vec{d}_D\ (x,u) = 0.4, \vec{d}_D\ (y,u) = 0.1, \vec{d}_D(x,y) = 0.5$.
Therefore $e(G_D) = \vec{d}_D(x,y) = 0.5$ and rad (G_D)=$\vec{d}_D(y,u) = 0.1$,
diam (G_D)=$\vec{d}_D(x,y) = 0.5$.
The central nodes are w and x. The peripheral nodes are u and v.

Theorem : 3.1

For any connected fuzzy diagraph G_D, rad $(G_D) \le$ diam $(G_D) \le 2$ rad (G_D).

Proof :

The inequality rad $(G_D) \le$ diam (G_D) is direct consequence of the definition. Since the smallest eccentricity. Cannot exceed of the largest eccentricity.

rad $(G_D) \le$ diam (G_D)

Then 2nd inequality vertices u & v in fuzzy diagraph $G_D: (V, \sigma_D, \mu_D)$.

$\vec{d}_D(u,v) =$ diam (G_D).

Furthermore,

Let w be a central vertex of G_D

$e_D(w)$ = rad (G_D)

$\vec{d}_D(u,v) \le \vec{d}_D(u,w) + \vec{d}_D(w,v)$

$\le$ rad (G_D) + rad (G_D)

≤ 2 rad (G_D)

$\vec{d}_D(u,v) \le 2$ rad (G_D)

rad $(G_D) \leq$ diam $(G_D) \leq 2$ rad (G_D).

Theorem : 3.2

If $G_D: (V, \sigma_D, \mu_D)$ is a self centered fuzzy digraph, then each node of G_D is eccentric.

Proof

Assume G_D is self centered and let u be any node of G_D.

Let v be an eccentric node of u. ie, u*=v Then $e(u) = \vec{d}_D(u,v)$.

Since G_D is self centered we have $e(v) = e(u)$

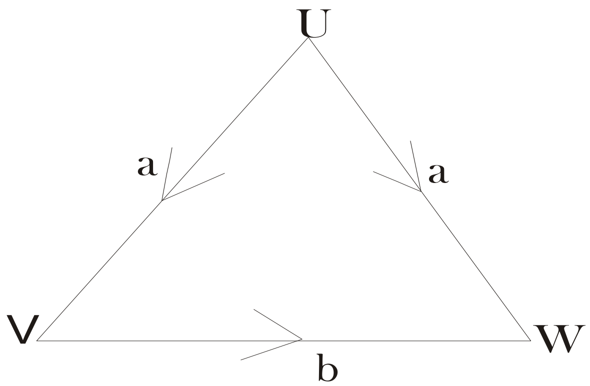
$$e(u) = \vec{d}_D(u,v) = e(v).$$

Which shows u is an eccentric node of v, i.e, v^* =u.

Hence the proof.

Proposition: 3.3

For any two real numbers a,b $such\ that\ o \leq a \leq b \leq 2a$. There exist a fuzzy digraph $G_D such\ that\ r(G_D) = a$ and $d(G_D) = b$.



Proof

$\vec{d}_D(u,v) = a$

$\vec{d}_D(u,w) = a$

$\vec{d}_D(v,w) = b$

Then, $e(u) = a, e(v) = b$

and $e(w) = b$

$\therefore r\ (G) = a$ and $d\ (G) = b$.

Theorem: 3.4

For every two node u and v in a connected fuzzy digraph $G_D : (V, \sigma_D, \mu_D)$, $ie\ (u) - e(v)1 \leq \vec{d}_D(u,v)$.

Proof

Assume without loss of generality $e(u) \geq e(v)$.

Let x be a node farthest from u.

i.e, $e(u) = \overrightarrow{d_D}(u,x) \leq d_D(u,v) + d_D\ (v,x)$.

By triangle inequality

$e(u) = \vec{d}_D(u,v) + e(v)$

Since $e(v) \geq \vec{d}_D(v,x)$

i.e, $o \leq e(u) - e(v) \leq \vec{d}_D(u,v)$

$|e(u) - e(v)| \leq \vec{d}_D(u,v)$.

Theorem: 3.5

The center of every connected fuzzy digraph $G_D : (V, \sigma_D, \mu_D)$, lies in a block of G^*.

Proof

Let $G_D : (V, \sigma_D, \mu_D)$ be a connected fuzzy digraph. Assume that the center of G_D does not lie in a block of $G_D *$. Then there exist a node v such that v is a cut node of $G_D *$.

Let G_{1D} and G_{2D} contain at least one central node of G_D. Let u be a node of G_D such that $\vec{d}_D(u,v) = e(v)$.

Let P_1 be a strong u-v path such that $\vec{d}_D(u,v) = L(P_1)$. length of P_1.

Then one of G_{1D} and G_{2D} contains no node in the path P_1 says G_{2D} contain no node of P_1.

Let w be a central node of G_D that belongs to G_{2D} and let be P_2 a strong v-w path such that $\vec{d}_D(v,w) = L(P_2)$. Length of P_2.

$\vec{d}_D(u,w) = L(P_1) + L(P_2)$.

Hence we have $e(w) > e(v)$ which contradicts w is a central node of G_D. Hence the center of every connected fuzzy digraph $G_D : (V, \sigma_D, \mu_D)$ lies in a block of G*.

4. MAXIMUM DISTANCE IN FUZZY DIGRAPHS

Definition: 4.1

Let $\sigma_D(v_i)$ and $\sigma_D(v_j)$ be a strong fuzzy digraph G_D. The maximum distance between $\sigma_D(v_i)$ and $\sigma_D(v_j)$ was defined by a fuzzy graph as the length of the shortest path between them. $m\vec{d}_D[\sigma_D(v_i)\ \ \sigma_D(v_j)]$= max $[\mu_D(v_i, v_j)]$

Definition: 4.2

Let $G_D : (V, \sigma_D, \mu_D)$ be a connected fuzzy digraph and let u be a node of G_D.

The m-eccentricity $me(G_D)$ of u is the maximum to a node. Farthest from u.

$me(G_D) = \max[\overrightarrow{d_D}(u,v)]$

Definition: 4.3

The m-radius mr(G_D) is the minimum m- eccentricity of the nodes.

mr(G_D) = min [e(u_d)]

Definition: 4.4

The m-diameter md(G_D) is the maximum eccentricity of nodes.

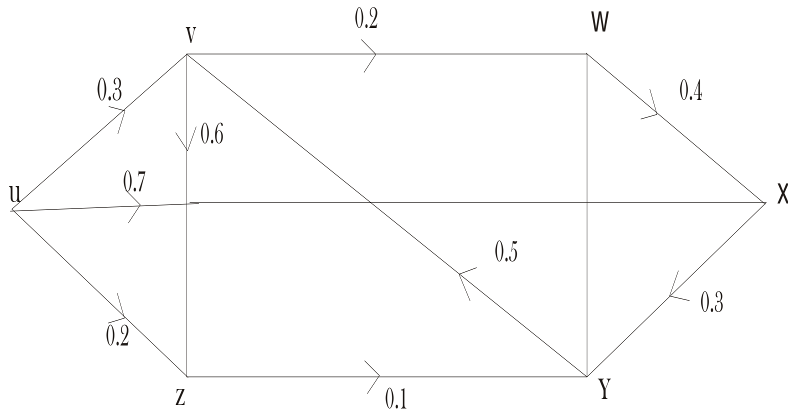
md(G_D) = max [me(G_D)]

Definition: 4.5

The fuzzy sub digraph by mc(G_D) denoted by m $< C(G_D) >$= H_D: (V τ_p γ_D) is called the m- center of G_D.

Definition: 4.6

A node G_D is a peripheral node if me(u_D)= md(G_D).



Example : 4.7

In figure m$\vec{d}$(u,v)= 0.8, m$\vec{d}$ (u,z)= 0.9, m$\vec{d}$ (v,z)= 0.6 m$\vec{d}$ (v,w)= 0.9, m$\vec{d}$ (w,x)= 0.4 m$\vec{d}$ (x,y)= 0.3, m$\vec{d}$ (u,x)= 0.9, m$\vec{d}$ (y,w)= 0.2, m$\vec{d}$(y,v)= 0.5.

Therefore me (G_D)= m$\vec{d}$ (v,w)= m$\vec{d}$(u,x)= 0.9 and mr((G_D)=m$\vec{d}$(y,w)=0.2.

Theorem : 4.1

For every strong fuzzy digraph G_D (V, σ_D, μ_D) the radius and diameter satisfy.

mr(G_D) ≤ md(G_D) ≤ 2 mr(G_D).

Proof:

mr(G_D) ≤ md(G_D) follows from the definition of radius and diameter.

Let w_D be a central node of G_D.

me (w_D) = mr(G_D)

Let u_D and v_D be two peripheral node of G_D.

me(u_D)= me(v_D)= md(G_D)

$\vec{d}_D$(u,v) ≤ $\vec{d}_D$ (u,w) + $\vec{d}_D$ (w,v)

md(G_D) ≤ mr(G_D) + mr(G_D)

≤ 2mr(G_D)

$mr(G_D) \leq md(G_D) \leq 2mr(G_D)$.

Theorem : 4.2

For every asymmetric fuzzy digraph G_D (V, σ_D, μ_D) There exists a strong asymmetric fuzzy digraph.

H_D: (V, τ_D, γ_D) *such that* $mc(H_D) \cong G_D$. Furthermore there exists such a fuzzy digraph H_D whose order exceeds that of G_D at most 4.

Proof

The result will allow us to show that the upper bound $mA(G_D) \leq 4$ cannot be improved in general for a vertex V in a fuzzy digraph G_D.

The out neighborhood $N^+(V)$ is the set of vertices G_D adjacent from V, while the in neighborhood $N^-(V)$ is the set of vertices adjacent to V.

Theorem: 4.3

Let G_D be an asymmetric fuzzy digraph with $mA(G_D)= 2$ and Let H_D be an asymmetric fuzzy digraph of minimum order with $m(CH_D) \cong G_D$. If $\vec{d}H_D(v,w_D)$ = m diamH_D then $\forall\ x \epsilon N^+(V)$ any $y \epsilon N^-(V)$ any shortest x-y path lies entirely in G_D.

Proof

Observe that mdiam H_D=max {$me(H_D)$}= $m\vec{d}_D$ (y,z) $x \epsilon V(H_D)$.

Where y,z ϵ $V(H_D)$- $V(G_D)$.

Since $mA(G_D)$=2 and $m\vec{d}_D(v,w)$ = mdiam (H_D).

It follows the $V(H_D)= V(G_D) \cup \{v, w\}$ and $meH_D(u)$= mrad $H_D \leq$ mdiam H_D $\forall$ u∈($V(G_D)$).

Suppose to the contrary that ∃ vertices x∈ N^+(V) and y∈ $N^-(w_D)$ such that a shortest x-y path P contains V or W_D.

If V_D lies on P then

$\vec{d}_D(x,y) = \vec{d}_D(x,y) + \vec{d}_D(v,y)$ [$(y,w) \in E(G_D)$; $(v, x) \in E(G_D)$]

It follows that,

$\vec{d}_D$ (y,w)= 1 and

$\vec{d}_D$ (x,y)> 1

$$\begin{aligned} \text{mdiam } H_D = \vec{d}_D(v,w) &\leq \vec{d}_D(v,y) + \vec{d}_D(y,w) \\ &\leq \vec{d}_D(x,v) + \vec{d}_D(v,y) \\ &\leq \vec{d}H_D(x,y) \\ &\leq m\vec{d}_D(x,y) \\ &\leq meH_D(x). \end{aligned}$$

Which contradicts the fact that $me_D(x)$ < mdiam H_D this proof is similar if w_D lies on a shortest x,y path in H_D.

Theorem : 4.4

An asymmetric fuzzy digraph G_D: (V, σ_D, μ_D) isomorphic to the m- periphery of some strong asymmetric fuzzy digraph H_D. iff

i) Every vertex of G_D has m- eccentricity 2 or.

ii) No vertex of G_D has m- eccentricity 2.

Proof:

The m-distance $m\vec{d}_D$ (v) of vertex V_D in a strong fuzzy digraph G_D is defined as,

$m\vec{d}_D(v)$= max { $m\vec{d}_D$ (v,u) }.

$u_D \in V(G_D)$

The m-median $mM(G_D)$ of G_D is the sub fuzzy digraph induced by those vertices having minimum m-distance.

The m-distance $m\vec{d}_D$ (H_{1D}, H_{2D}) between two sub fuzzy digraph H_{1D} and H_{2D} in a fuzzy digraph H_D is defined by $\vec{d}_D$ (H_{1D}, H_{2D}) = min{$m\vec{d}_D$ (u,v)} u ∈ H_{1D} and V ∈ H_{2D}.

5. SUM DISTANCE IN FUZZY DUGRAPHS

Definition: 5.1

The sum distance $S\vec{d}_D$ [($\sigma_D(V_i)$, ($\sigma_D(V_j)$)] between two nodes $\sigma_D(V_i)$ and $\sigma_D(V_j)$ in a fuzzy digraph G_D as the length of the shortest path between then,

$S\vec{d}_D$ [($\sigma_D(V_i)$, ($\sigma_D(V_j)$)]= Min {μ_D (V_i, V_j)}.

Definition: 5.2

The S- eccentricity $S(e(G_D))$ of G_D is the sum distance to a node farthest from u_D.

$Se(G_D)$= Max [$\vec{d}_{SD}$ (u,v): V ∈ V_D].

Definition : 5.3

The S- radius $r(G_D)$ is the minimum eccentricity of the nodes $Sr(G_D)$ = min $Se(G_D)$.

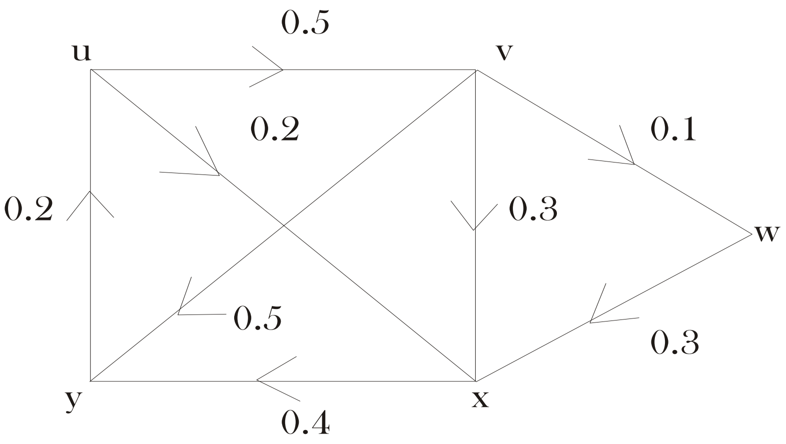
The S- diameter $D(G_D)$ is the maximum eccentricity of the nodes.

Sd (G_D)= max Se(G_D)

Definition: 5.4

The fuzzy sub digraph induced by C (G_D) denoted by$< C(G_D) >$ H_D: (V, τ_D, γ_D) is called the center of G_D.

A node u_D is a peripheral node if Se(u_D)= d(G_D).



Example: 5.5

In figure S$\vec{d}_D$ (u,v) = 0.5, S$\vec{d}_D$ (u,x)= 0.2, S$\vec{d}_D$ (v,x)= 0.3, S$\vec{d}_D$ (u,x)= 0.2, S$\vec{d}_D$ (w,x)= 0.3, S$\vec{d}_D$ (v,w)= 0.1, S$\vec{d}_D$ (x,y)= 0.4, S$\vec{d}_D$ (y,u)= 0.2, S$\vec{d}_D$ (v,y)= 0.5. Therefore e(u)=0.5, e(v)=0.4, e(w)=0.3, e(x)=0.4, e(y)= 0.2. Se(G_D)= S$\vec{d}_D$ (u,v)= 0.5. Sr(G_D)= e(y)= 0.2.

Theorem: 5.1

For every strong fuzzy digraph G_D: (V, σ_D, m_D) the radius and diameter satisfy.

Sr(G_D) ≤ Sd(G_D) ≤ 2 Sr(G_D).

Proof

Sr(G_D) ≤ Sd(G_D) follows from the Definition: of radius and diameter.

Let w_D be a central node of G_D.

Se(w_D)= Sr(G_D)

Let u_D and v be two peripheral node of G_D.

Se(u_D) = Se(V_D) = Sd(G_D)

$\vec{d}_{SD}$ (u,v) ≤ $\vec{d}_{SD}$ (u,w) + $\vec{d}_{SD}$ (w,v)

$S\vec{d}$(G_D) ≤ Sr(G_D) + Sr(G_D)

≤ 2Sr(G_D)

Sr(G) ≤ Sd(G_D) ≤ 2Sr(G_D).

Theorem: 5.2

For every asymmetric fuzzy digraph G_D: (V, σ_D, μ_D). There exists a strong asymmetric fuzzy digraph H_D: (V, τ_D, γ_D) such that SC(H_D) ≅ G_D. Furthermore there exists such a fuzzy digraph H_D whose order exceeds that of G_D by at most 6.

Proof

For an asymmetric fuzzy digraph G_D: (V, σ_D, μ_D) we define the sum appendage number A(G_D) of G_D is the minimum number of vertices that must added to G_D to produce a strong asymmetric fuzzy digraph H_D.

Whose S-center is isomorphic to G_D.

SA(G_D) ≤ 6 for every asymmetric fuzzy digraph G_D.

Theorem: 5.3

An asymmetric fuzzy digraph G_D: (V, σ_D, μ_D) is isomorphic to the S-periphery of some strong asymmetric fuzzy digraph H_D: (V, τ_D, γ_D) iff,

i) Every vertex G has S- eccentricity 3 or
ii) No vertex of G has S- eccentricity 3

Proof

The S- periphery SP(G_D) of a strong asymmetric fuzzy digraph G_D is the sub fuzzy digraph induced by those vertices of maximum e- eccentricity.

It should not be noted there exists an asymmetric fuzzy digraph that is isomorphic to the s-periphery of some strong asymmetric fuzzy digraph but not to the m- periphery of strong asymmetric fuzzy digraph.

CONCLUSION

Thus, we have discussed the concept on distance is digraphs. The idea of distance in fuzzy digraph is introduced the concepts of eccentricity, radius, diameter etc., with examples. Discussed some of the properties on maximum distance in fuzzy digraphs and the sum distance in fuzzy digraphs.

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