REVIEW OF LYAPUNOV'S CRITERIA: A WAY TO ANALYZE THE STABILITY OF LINEAR AND NONLINEAR SYSEMS

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Abstract: The main aim of this paper is to review the approach used for stability criteria using direct and indirect Lypunov's methods for linear and nonlinear system. There are various techniques rather computational techniques for stability analysis of linear or nonlinear dynamical system but Lyapunov's method are still highly useful for stability analysis. Although stability of linear time invariant (LTV) system can be examined by lyapunov direct or indirect method both but as far as concern of nonlinear system lyapunov indirect or first method is not feasible always to check the stability. For direct Lyapunov method we require proper lyapunov function to check the stability so there are various techniques used by the researcher to construct lyapunov function like Krasovskii approach, Positive Dimensional Polynomial System approach, Variable gradient method etc.

Keywords: Autonomous linear differential equation, Lyapunov function, Nonlinear time invariant systems, Stability

1.Introduction

Alexander Mikhailovich Lyapunov did a lot of work on stability analysis of linear and nonlinear system. He proved various theorems step by step on the different types of stability which are still highly useful. The study of stability basically to analyze whether a system returns to its original position or remains near to the original position after small perturbation or not. There are generally two Lyapunov's methods known as indirect method (first method) and direct method (second method). Indirect method is based on characteristic value approach while the direct method is based on Lyapunov's function. *This powerful idea originated in Lyapunov's mind from his knowledge of astronomical problems and the earlier work of Lagrange, Laplace, Dirichlet, and Liouville (Fuller 1992)*[1].

2. Basic Terminology

There are some basic terms and definitions required to understand the concern topic properly as follow.

2.1 Def. A differential equation of the form $\frac{dx}{dt} = f(x)$

Where x is function of time, is called autonomous differential equation as $\dot{x}$ is not explicitly depends upon time t, which is linear or non linear differential equation further depends upon the function $f(x)$.

2.2 Def. For an autonomous system $\frac{dx}{dt} = f(x)$, Equilibrium points or rest points are those values of x for which $f(x) = 0$.

2.3 Def. A function $f: A \to R$ is said to be positive definite in A if it satisfied the following conditions :

(i) $0 \in A \; and \; f(0) = 0$

(ii) $f(x) > 0 \; \forall \; x \in A$

2.4 Def. An equilibrium point x_α is stable , provided for each $\varepsilon > 0 \; \exists \; \delta(\varepsilon, t) > 0 \;\; s.t.$

$$\|x_0 - x_\alpha\| < \delta(\varepsilon, t) \quad \Rightarrow \quad \|x_t - x_\alpha\| < \varepsilon \qquad \forall \quad t \geq t_0$$

Moreover in addition to above condition if we have $\lim_{t\to\infty} x_t = x_\alpha$ then equilibrium point is called asymptotic stable. On the other hand if δ is independent of time then it becomes uniform asymptotic stable.For exponential stability $\exists \; \beta, \gamma > 0 \; s.t$.

$$\|x_t - x_\alpha\| \leq \beta\|x_0 - x_\alpha\|e^{-\gamma t} \;\; \forall \, t > 0 \quad whenever \; \|x_0 - x_\alpha\| < \delta.$$

2.4 ***Lyapunov Function****: For general dynamical system* $\frac{dX}{dt} = f(X)$ *,a lyapunov function* $V(X)$ *is a function which satisfied*

$$\nabla V(X). f(X) \leq 0.$$

There is no simple way to determine whether a Lyapunov function exists for a given dynamical system, or, if it does exist, what the Lyapunov function is.[2]

3. Main Results

3.1 Theorem (Stability) : Let $X = 0$ be an equilibrium point of the dynamic system $\frac{dX}{dt} = f(X)$ & f: D→ R^n $and \; V: D \to R \;\; Where \; D \subset R^n$, be a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(X) > 0 \; in \; D - \{0\}$

(iii) $\dot{V}(X) \leq 0 \; in \; D - \{0\}$

Then "$X = 0$" is Stable.

3.2 Theorem (Asymptotically Stability) : Let $X = 0$ be an equilibrium point of the dynamic system $\frac{dX}{dt} = f(X)$ & f: D→ $R^n \; and \; V: D \to R \; Where \; D \subset R^n$, be a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(X) > 0 \; in \; D - \{0\}$

(iii) $\dot{V}(X) < 0 \ in \ D - \{0\}$

Then "$X = 0$" is Asymptotically Stable.

3.3 Theorem (Globally Asymptotically Stability) : Let $X = 0$ be an equilibrium point of the dynamic system $\frac{dX}{dt} = f(X)$ & $f: R^n \to R^n \ and \ V: R^n \to R$ be a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $V(X) > 0 \ in \ R^n - \{0\}$

(iii) $V(X)$ is radially unbounded i.e. $\|X\| \to \infty \Rightarrow V(X) \to \infty$

(iv) $\dot{V}(X) < 0 \ in \ R^n - \{0\}$

Then "$X = 0$" is Globally Asymptotically Stable.

3.4 Theorem (Instability) : Let $X = 0$ be an equilibrium point of the dynamic system $\frac{dX}{dt} = f(X)$ & $f: R^n \to R^n \ and \ V: R^n \to R$ be a continuously differentiable function such that

(i) $V(0) = 0$

(ii) $\exists X_0 \in R^n, arbitrary \ close \ to \ X = 0 \ s.t. V(X) > 0$

(iii) $\dot{V} > 0 \ \forall \ X \in U \ Where \ U = \{X \in R^n | \|X\| < \varepsilon \ \& \ V(X) > 0\}$

Then "$X = 0$" is Unstable.

3.5 Theorem: Consider a Linear system$\dot{X} = AX$, with eigen values$\lambda_i, i = 1,2, \dots \dots, n$

(i) If the system is stable, then $Re(\lambda_i) \le 0, i = 1,2, \dots \dots, n$

(ii) If either $Re(\lambda_i) < 0, i = 1,2, \dots \dots, n \ or \ if \ Re(\lambda_i) \le 0, i = 1,2, \dots \dots, n$ and there is no zero repeated characteristic values; then the system is uniformly stable.

(iii) The system is asymptotically stable iff $Re(\lambda_i) < 0, i = 1,2, \dots \dots, n$

(iv) If $Re(\lambda_i) > 0 \ for \ any \ i$, the solution is unstable.

Lyapunov's First method

The system of nonlinear first order equation $\frac{dX}{dt} = f(X)$ can be linearized in the form

$$\dot{X} = JX$$

$Where \ J \ is \ Jacobian \ matrix \ J \ with \ the \ (i,j) entry \ J_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]_{x=0}$ and X=0 is the equilibrium point.[1]

If the matrix J has all its characteristic values either negative or with negative real parts, so that the linearized equation is asymptotically stable, then so will be the original nonlinear equation. This is a sufficient condition not necessary.

Lyapunov's Second method

For the first order nonlinear differential equations

$$\dot{x_i} = f_i(x_1, x_2, \dots \dots, x_n)$$

having origin X=0 as equilibrium point (if not then by shifting of origin we can make it).We have to find a positive definite function $V = V(X) s.t.$

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x_1, x_2, \dots, x_n)$$

is always negative except at X=0. If such happened then equilibrium point is asymptotically stable.

Example: Consider the nonlinear system as

$$\dot{x_1} = x_2 x_3 - 3x_3$$
$$\dot{x_2} = x_1 x_3$$
$$\dot{x_3} = x_1 - x_1 x_2$$

Clearly $(0,0,0)$ be the equilibrium point of above system.

We take Lyapunov's function as $V(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 > 0 \ for \ X \ne 0 \ and \ V(0) = 0.$

$$\nabla V . f = 0 \ \forall \ X$$

Therefore equilibrium point is stable but not asymptotically. Phase portraits of solution of above equation is given by $V(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 = k^2 \ , where \ k \epsilon \ R.$

Example: Consider the nonlinear system as

$$\dot{x_1} = -x_1 + (x_2 - 3)x_3$$
$$\dot{x_2} = -x_2 + x_1 x_3$$
$$\dot{x_3} = -x_3 + x_1(1 - x_2)$$

Clearly $(0,0,0)$ be the equilibrium point of above system.

We take Lyapunov's function as $V(x_1, x_2, x_3) = \frac{1}{7}x_1^2 + \frac{2}{7}x_2^2 + \frac{3}{7}x_3^2 > 0 \ for \ X \ne 0 \ and \ V(0) = 0.$

$$\nabla V . f = -\frac{2}{7}(x_1^2 + 2x_2^2 + 3x_3^2) < 0 \quad \forall \ X \ne 0$$

Therefore equilibrium point $(0,0,0)$ is asymptotically stable. Phase portraits of solution of above equation is given by as $V(x_1, x_2, x_3) = \frac{1}{7}x_1^2 + \frac{2}{7}x_2^2 + \frac{3}{7}x_3^2 = k^2 \ , where \ k \epsilon \ R.$ which are the closed surface of ellipsoid around the origin.

Conclusion:

Although there is no proper method or rule for selection of lyapunov function but still these methods play a great role in the stability analysis of linear and non linear differential equations. Moreover stability analysis has great importance in the case of theory of impulsive differential equations. The impulsive differential equations involving impulse effects appear as a natural description of observed evolution phenomena of several real world problems. Therefore Lyapunov methods of stability are directly or indirectly useful to understand and also to find solution of real world problems.

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