FIXED POINT FOR RATIONAL TYPE ALMOST CONTRACTION IN PARTIALLY ORDERED METRIC SPACE

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Abstract : ***In this paper, we establish some fixed point theorems for a mapping with monotone property satisfy a rational type almost contraction in the partially ordered metric spaces. The results presented in the paper generalize and extend some known results in the existing literature and examples are also given to validate our results***

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1. Introduction and preliminaries

Metric fixed point theory is widely recognized to have been originated in the work of S.Banach[3],where he proved the famous contraction mapping principle. The basic unit of analysis in order theory is the binary relation, it is well known that a relation $\mathfrak{R}$ on a set X is a subset of $X \times X$. We denoted $(x, y) \in \mathfrak{R}\ by\ x\, \mathfrak{R}\, y$. In "order" on a set X is a relation on X satisfy some additional conditions, order relation are usually denoted by " $\leq$ "

Definition 1.1 A partially ordered set is a set X with a binary operation denoted by $(X, \leq)$ such that for $p, q, r \in X$

(i) $p \leq p\ (reflexive)$

(ii) $p \leq q\ and\ q \leq p \Rightarrow p = q\ (anti\ symmetry)$

(iii) $p \leq q\ and\ q \leq r \Rightarrow p \leq r\ (transitivity)$

Khan et. al[18] initiated the use of control function that alter distance between two points in metric space which they called an alternating distance function.

Definition 1.2[18] A function $\phi: [o, \infty) \to [o, \infty)$ is called an alternating distance function if the following properties are satisfied:

(i)ϕ is a continuous function.

(ii)ϕ is monotonically increasing function.

(iii)$\phi(x) = 0 \Leftrightarrow x = 0$

In our result we will use the following class of function $\Phi = \{\phi: [0, \infty) \to [0, \infty) : \phi$ an alternating distance function$\}$

In[13] Dass and Gupta represented the following fixed point theorem

Theorem 1.3 [13] Let (X, d) be a Complete metric space and $T: X \to X$ a mapping Such that there exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(Tx, Ty) \leq \alpha \frac{d(y,Ty)(1 + d(x,Tx))}{1 + d(x,y)} + \beta\, d(x, y) \qquad \forall\, x, y \in X \qquad (1.1)$$

Then T has a unique fixed point

In [6] Cabrera et. al proved the above theorem in the context of partially ordered metric space.

Definition1.4 [6] Let $(X, \leq)$ be a partially ordered set and $T: X \to X$ The mapping T is said to be nondecreasing if for all $x, y \in X, x \leq y\ implies\ Tx \leq Ty$ and nonincreasing if for all

$$x, y \in X, x \leq y\ implies\ Tx \geq Ty$$

Theorem 1.5 [6] Let $(X, \leq)$ be a partially ordered set and Suppose that there exists a metric d in X such that (X, d) is a Complete metric space. Let $T: X \to X$ be a Continuous and nondecreasing mapping such that 1.1 is satisfied for all $x, y \in X\ with\ x \leq y$. If there exists $x_0 \in X$ Such that $x_0 \leq Tx_0$, then T has a fixed point.

Theorem 1.6 [6] Let $(X, \leq)$ be a partially ordered set and Suppose that there exists a metric d in X such that (X, d) is a Complete metric space. Assume that $\{x_n\}$ nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$ for all $n \in N$. Let $T: X \to X$ be a nondecreasing mapping such that 1.1 is satisfied for all $x, y \in X\ with\ x \leq y$. If there exists $x_0 \in X$ Such that $x_0 \leq Tx_0$, then T has a fixed point.

Fixed point theorem for contractive type conditions satisfying rational inequalities in metric spaces have been developed in a number of works ([8],[9],[10],[14],[15],[16],[19]). The concept of almost contractions were introduced by Berinde ([4],[5]).

Definition 1.7 [4] Let (X, d) be metric space. a mapping $T: X \to X$ is called an Almost Contraction if there exists $k \in (0, 1)$ and some $L \geq 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq kd(x, y) + Ld(y, Tx)$$

It was shown in [5] that any strict contraction, the Kannan[17] and Zamfirescu[21] mappings, as well as a large class of quasi contraction are all almost contraction and generalizations were further considered in several work like([1],[2],[11],[12],[7]).The purpose of this paper is to study about fixed point of a class of mappings satisfying a rational type almost contraction in ordered metric space. Our results are supported with examples.

2. Main Results

Theorem 2.1 Let $(X, \leq)$ be a partially ordered set and Suppose that there exists a metric d in X such that (X, d) is a Complete metric space. Let $T: X \to X$ be a Continuous.T is monotone nondecreasing mapping and $\phi \in \Phi$ and

$$d(Tx, Ty) \leq \quad k\phi \max\{\frac{d(x, Ty)d(y, Tx)(d(x, Tx) + d(y, Ty)}{1 + d(x, y)}, d(x, y)\}$$

$$+L\, min\big(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big) \tag{2.1}$$

where $0 < k < 1$ and $L \geq 0$ for $x, y \in X$ with $x \geq y, x \neq y$.If there exists $x_0 \in X$ Such that $x_0 \leq Tx_0$ then T has a fixed point.

Proof. If $x_0 = Tx_0$ then we have the same Results. Suppose that $x_0 < Tx_0$. Then we construct a sequence $\{x_n\}$ in X such that

$$x_n \leq Tx_n \qquad \forall\, n = 0, 1, 2, 3 \ldots\ldots\ldots\ldots\ldots \tag{2.2}$$

Since T is nondecreasing mapping, we obtain by induction that

$$x_0 \leq Tx_0 = x_1 \leq Tx_1 = x_2 \leq \ldots\ldots\ldots \leq Tx_{n-1} = x_n \leq Tx_n \ldots\ldots \tag{2.3}$$

If there exists some $k \in N$ such that $x_{k+1} = x_k$ then by $x_{k+1} = x_k = Tx_k$ that is x_k is a fixed point of T and the proof is finished. So we suppose that $x_{n+1} \neq x_n \ \ \forall \ \ n \in N$ Since $x_n \leq x_{n+1}$ $\forall\, n \in N$.we set $x = x_n$ $and\ y = x_{n+1}$ in 2.1 we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$

$$\leq k\,\phi \max\{\frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)\big(d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\big)}{1 + d(x_n, x_{n+1})}, d(x_n, x_{n+1})\}$$

$$+ L\, min(d\,(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n))$$

$$\leq k\phi\{\max\,(\frac{d(x_n, x_{n+2})d(x_{n+1}, x_{n+1})(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, d(x_n, x_{n+1})\}$$

$$+ L min\big(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\big) \leq$$

$$k\,\phi\,\{max(\,0\,, d(x_n, x_{n+1}))\}$$

$$\leq \ k\,\phi\ d(x_n, x_{n+1})$$

$$d(x_{n+1}, x_{n+2}) \leq \ k\ d(x_n, x_{n+1})$$

$$\leq \ k^2\ d(x_{n-1}, x_n)$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$\leq \ k^{n+1}\, d(x_0, x_1)$$

Then $for\ any\ \ m > n$

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1}, x_m) \leq (k^n + k^{n+1} + \ldots.. + k^{m-1})d(x_0, x_1)$$

$$\leq \frac{k^{n+1}}{1-k}\, d(x_0, x_1)$$

taking $n \to \infty$ implies that $\{x_n\}$ is a cauchy sequence. From the completeness of X there exists $x \in X$ such that

$$x_n \to x \tag{2.4}$$

The continuity of T implies that $Tx = \lim_{n\to\infty} Tx_n$

$$= \lim_{n\to\infty} x_{n+1}$$

$$Tx = x$$

i.e. x is fixed point of T.

Uniqueness: Let x and y be the fixed point of T such that $x \neq y$ then $Tx = x$ and $Ty = y$.

Now $\quad d(x, y) = d(Tx, Ty)$

$$\leq \ k\phi\{max\left(\frac{d(x, Ty)d(y, Tx)\big(d(x, Tx) + d(y, Ty)\big)}{1 + d(x, y)}, d(x, y)\right)\}$$

$$+ L\, min(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

$$\leq \ k\phi\{max\left(\frac{d(x,y)d(y,x)\big(d(x,x)+d(y,y)\big)}{1+d(x,y)}, d(x, y)\right)\}$$

$$+ L\min\big(d(x, x), d(y, y), d(x, y), d(y, x)\big)$$

$$\leq \ k\,\phi\,\{max(\,0\,, d(x, y))\} + L(0)$$

$$\leq \ k\, d(x, y)$$

which is only possible if $d(x, y) = 0 \Rightarrow x = y$

In our next theorem we relax the Continuity assumption of the mapping T in theorem 2.1 and find the fixed point.

Theorem 2.2 Let $(X, \leq)$ be a partially ordered set and Suppose that there exists a metric d in X such that (X, d) is a Complete metric space. Let $T: X \to X$ be a non-decreasing mapping such that $\forall\, x, y \in X\ with\ x \leq y$. 2.1 is satisfied where the condition of k and L are Same as in theorem 2.1. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$ then T has a fixed point.

Proof: We take the same sequence $\{x_n\}$ as in the proof of theorem 2.1 arguing like in the proof $\{x_n\}$ is a non-decreasing which satisfy 2.4 that is $x_n \to x$ as $n \to \infty$.

Then by the conditions of the theorem $x_n \leq x \quad \forall\ n \in N$ apply 2.1 we have

$$d(x_{n+1}, Tx) = d(Tx_n, Tx)$$

$$\leq \ k\phi\left\{\max(\frac{d(x_n,Tx)d(x,Tx_n)\big(d(x_n,Tx_n) + d(x,Tx)\big)}{1+d(x_n,x)}, d(x_n, x_{n+1})\right)\}$$

$$+ min\,\{d(x_n, Tx_n), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\}$$

$$\leq \ k\,\phi\{\,max(\frac{d(x_n,Tx)d(x,x_{n+1})(d(x_n,x_{n+1})+d(x,Tx))}{1+d(x_n,x)}, d(x_n, x_{n+1})\}$$

$$+ min(d(x_n, x_{n+1}), d(x, Tx), d(x_n, Tx), d(x, x_{n+1}))$$

Taking the limit as $n \to \infty$ in the above inequality and using 2.4 we have

$$d(x, Tx) = 0$$
$$Tx = x$$

that is x is a fixed point of T.

Example 3.1 Let $X = [0,1]$ partial order $\preccurlyeq$ defined as $x \preccurlyeq y$ if and only if $x \leq y$. Let the Real valued metric d be given as $d(x, y) = |x - y|$

and let $T: X \to X$ be defined as follows $Tx = \frac{x}{2}$ and $\phi(t) = t$.

Here all the conditions of theorem 2.1 are satisfied and 0 is a fixed point T.

Example 3.2 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with usual partial order $\leq$ Let us consider the Real valued metric d be given as

$$d(x, y) = \begin{cases} \max\{x, y\}; & \text{if } x \neq y \\ 0; & if\ x = y \end{cases}$$

and let $T: X \to X$ be defined as follows

$$T(x) = \begin{cases} \frac{x}{2}; & if\ x\ is\ even \\ \frac{x+1}{2}; & if\ x\ is\ odd \end{cases}$$

$and\ \ \phi(t) = t.$

Here all the conditions of theorem 2.2 are satisfied and x=1 is a fixed point.

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