A NEW ONE-STEP RATIONAL METHOD FOR SOLVING STIFF AND NON-STIFF DELAY DIFFERENTIAL EQUATIONS

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Abstract: This paper presents the one-step fourth order rational method to solve the Delay differential equations (DDEs) by using interpolating function which consists of rational function. The delay argument is approximated using Lagrange interpolation. The stability polynomial of this method and the corresponding stability region are obtained. The applicability of this method has been demonstrated by numerical examples of stiff and non-stiff DDEs with constant delay, time dependent delay and state dependent delays. The numerical results are compared with the theoretical solution.

Index Terms **- Lagrange interpolation, One-Step Method, Rational function, Stability polynomial and Stability region, Stiff and Non-Stiff Delay differential equations.**

I. INTRODUCTION

Delay differential equations occurs in chemical kinetics [1], population dynamics [2], and management systems [3] and in several areas of science and engineering. Recently there has been a growing interest in the numerical solutions of stiff and non-stiff DDEs. Some of the notable numerical methods are Runge Kutta Method [4], Homotopy Perturbation method [5], Adomian Decomposition Method [6], Block method [7], Difference method [8] and Parallel two-step ROW method [9].

Most of the models in differential equations are 'stiff' in nature. For solving stiff equations, the step size is taken to be extremely small. Also, many problems may be stiff in some intervals and non-stiff in others. Therefore, we need an efficient technique to be suitable for stiff and non-stiff problems. Several one-step numerical techniques have been developed for the solution of first order differential equations by means of interpolating functions. These works can be referred in [10-13].

In this paper we present a new one-step rational method for solving stiff and non-stiff DDEs with constant, time and state dependent delays. In this method, the solution is represented as a rational function. The organization of this paper is as follows: In section 2, the derivation of the new method is described. In section 3, the stability analysis of the new one-step method is investigated. Stability region is also deduced. In section 4, numerical illustrations of DDEs are provided to demonstrate the efficiency of this method.

II. THE DERIVATION OF THE NEW ONE-STEP RATIONAL METHOD

Consider the first order DDEs with delay τ,

$$y'(t) = f\Big(t, y(t), y\big(t-\tau(t, y(t))\big)\Big), \qquad t > t_0$$

$$y(t) = \Phi(t), \qquad t \le t_0 \qquad (1)$$

where $\Phi(t)$ is the initial function.

Let us assume that the analytical solution $y(t_{n+1})$ to the initial value problem (1) can be locally represented in the interval $[t_n, t_{n+1}]$, $n \ge 0$ by rational function as

$$y_{n+1} = A + \frac{a_1 t_n}{\sum_{j=0}^{3} b_j t_n^j} \qquad (2)$$

$$y_{n+1} = A + \frac{a_1 t_n}{b_0 + b_1 t_n + b_2 t_n^2 + b_3 t_n^3} \qquad (3)$$

where a_1, b_0, b_1, b_2 and b_3 are undetermined coefficients.

By taking k = 3 and $b_0 = 1$,

$$y_{n+1} = A + a_1 t_n [1 + b_1 t_n + b_2 t_n^2 + b_3 t_n^3]^{-1} \tag{4}$$

Using Binomial expansion in (4), we get

$$y_{n+1} = A + a_1 t_n \left(1 + (-1)[b_1 t_n + b_2 t_n^2 + b_3 t_n^3] + \frac{(-1)(-2)}{2!}[b_1 t_n + b_2 t_n^2 + b_3 t_n^3]^2 + \frac{(-1)(-2)(-3)}{3!}[b_1 t_n + b_2 t_n^2 + b_3 t_n^3]^3\right)$$

$$y_{n+1} = A + a_1 t_n\left(1 - b_1 t_n - b_2 t_n^2 - b_3 t_n^3 + {b_1}^2 t_n^2 + 2 b_1 b_2 t_n^3 \quad + 2 b_1 b_2 t_n^3 + {b_1}^3 t_n^3\right)$$

$$y_{n+1} = A + a_1 t_n - a_1 b_1 t_n^2 + \left(-b_2 a_1 + {b_1}^2 a_1\right) t_n^3 + \left(-b_3 a_1 + 2 b_1 b_2 a_1 + {b_1}^3 a_1\right) t_n^4 \tag{5}$$

Expressing the left hand side of (5) in terms of Taylor's series expansion,

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{(iv)} \tag{6}$$

By making the above expression to agree term by term for each parameter, we get

$$A = y_n \tag{7}$$

$$a_1 = \frac{h y_n'}{t_n} \tag{8}$$

$$b_1 = \frac{-h y_n''}{2 y_n' t_n} \tag{9}$$

$$b_2 = \frac{h^2\left(3(y_n'')^2 - 2 y_n' y_n''' t_n^3\right)}{12 (y_n')^2 t_n^2} \tag{10}$$

$$b_3 = \frac{-h^3 y_n^{(iv)} t_n}{2 y_n'} - \frac{3 h^3 (y_n'')^3}{24 (y_n')^3 t_n^3} + \frac{4 h^3 y_n'' y_n'''}{24 (y_n')^2} \tag{11}$$

Substituting (7), (8), (9), (10) and (11) in (3), we get

$$y_{n+1} = y_n + \left[\frac{24 h (y_n')^4}{24 (y_n')^3 - 12 h (y_n')^2 y_n'' + h^2\left(6 y_n' (y_n'')^2 - 4 (y_n')^2 y_n'''\right) t_n^3 + h^3\left(4 y_n' y_n'' y_n''' t_n^3 - 12 (y_n')^2 y_n^{(iv)} t_n^4 - 3 (y_n'')^3\right)}\right] \tag{12}$$

Eqn. (12) is the new one-step rational numerical technique.

This one-step method can also be extended to solve DDEs with multiple delays. In this paper, Lagrange interpolation is used to approximate the delay argument.

III. ANALYSIS OF THE ONE-STEP RATIONAL METHOD

A general one-step method is given in form

$$y_{n+1} = y_n + \phi(t_n, y_n; h),$$

where $\phi(t_n, y_n; h)$ is called as incremental function of the method.

By rearranging (12) and using Taylor's series, we obtain the incremental function as,

$$\phi(t_n, y_n; h) = y_n' + \frac{h}{2} y_n'' + \frac{h^2}{6} y_n''' + \frac{h^3}{24} y_n^{(iv)} + O(h^4) \tag{13}$$

3.1 Order and Consistency of the Method

Definition:

A numerical scheme with an incremental function $\phi(t_n, y_n; h)$ is said to be consistent with the initial value problem (1) if

$$\phi(t_n, y_n; 0) = f(t_n, y_n).$$

Using the above definition, we see that our one-step rational method is consistent. Also, by virtue of Taylor series it is found that this one-step rational method given by (12) is of order four.

3.2 Stability Polynomial of the Method

Here we consider a commonly used linear test equation with a constant delay $\tau = mh$, where m is a positive integer,

$$y'(t) = \lambda y(t) + \mu y(t - \tau), \quad t > t_0$$

$$y(t) = \phi(t), \qquad t \le t_0$$

where $\lambda, \mu \in C$, $\tau > 0$ and ϕ is continuous.

A slight arrangement of (12), we obtain that

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + \frac{h^4}{24}y_n^{(iv)} \tag{14}$$

This implies,

$$y_{n+1} = y_n + h(\lambda y_n + \mu y(t_n - \tau)) + \frac{h^2}{2}\left(\lambda y'_n + \mu y'(t_n - \tau)\right) + \frac{h^3}{6}\left(\lambda y''_n + \mu y''(t_n - \tau)\right) + \frac{h^4}{24}(\lambda y'''_n + \mu y'''(t_n - \tau)) \tag{15}$$

Here Lagrange interpolation is used to approximate the delay term.

$$y(t_n - mh) = y(t_{n-m}) = \sum_{l=-r_1}^{s_1} L_l(c_i) y_{n-m+l}$$

with

$$L_l(c_i) = \prod_{j_1=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \qquad j_1 \neq l \quad and\ r_1, s_1 > 0$$

The new one-step method is applied to DDE with (1), with constant delay $\tau = 1$.

Now $y(t_n - \tau) = \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$,

$y'(t_n - \tau) = \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}$ and

$$y''^{(t_n-\tau)} = \lambda\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\right) + \mu\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l}\right)$$

$$y'''(t_n - \tau) = \lambda\left(\lambda\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\right) + \mu\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l}\right)\right)$$
$$+\mu\left(\lambda\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l}\right) + \mu\left(\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-4m+l}\right)\right) \tag{16}$$

Substituting (16) in (15), we get

$$y_{n+1} = y_n + h\left(\lambda y_n + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\right) + \frac{h^2}{2}\left(\lambda^2 y_n + 2\lambda\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu^2 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\right)$$
$$+\frac{h^3}{6}\left(\lambda^3 y_n + 3\lambda^2\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + 3\lambda\mu^2 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} + \mu^3 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l}\right)$$
$$+\frac{h^4}{24}\left(\lambda^4 y_n + 4\lambda^3\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + 6\lambda^2\mu^2 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} + 4\lambda\mu^3 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l} + \mu^4 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-4m+l}\right)$$

$$y_{n+1} = y_n + \lambda h y_n + \frac{\lambda^2 h^2}{2} y_n + \frac{\lambda^3 h^3}{6} y_n + \frac{\lambda^4 h^4}{24} y_n + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\left(\mu h + \mu\lambda h^2 + \frac{\lambda^2\mu h^3}{2} + \frac{\lambda^3\mu h^4}{6}\right)$$
$$+ \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\left(\frac{\mu^2 h^2}{2} + \frac{\lambda\mu^2 h^3}{2} + \frac{\lambda^2\mu^2 h^4}{4}\right) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-3m+l}\left(\frac{\mu^3 h^3}{6} + \frac{\lambda\mu^3 h^4}{6}\right)$$
$$+ \sum_{l=-r_1}^{s_1} L_l(c) y_{n-4m+l}\left(\frac{\mu^4 h^4}{24}\right)$$

Let $\alpha = \lambda h$ and $\beta = \mu h$. Then the above equation becomes,

$$y_{n+1} = y_n\left(1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!}\right) + \left(\beta\left(1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\right)\right) \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$$

$$+\left(\beta^2\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)\right)\sum_{l=-r_1}^{s_1} L_l(c)y_{n-2m+l}+\left(\beta^3\left(\frac{1}{6}+\frac{\alpha}{6}\right)\right)\sum_{l=-r_1}^{s_1} L_l(c)y_{n-3m+l}+\left(\frac{\beta^4}{24}\right)\sum_{l=-r_1}^{s_1} L_l(c)y_{n-4m+l}$$

To obtain the stability region of the method, the delay term is approximated using five points Lagrange interpolation. By putting $n-m+l=0$, $n-2m+l=0$, $n-3m+l=0$, $n-4m+l=0$ and by taking $l=-1,0,1,2,3$, the stability polynomial will be in the standard form. The recurrence is stable if the zeros ζ_i of the stability polynomial

$$\begin{aligned} S(\alpha,\beta:\zeta) = \zeta^{n+1} &- \left(1+1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\right)\zeta^n \\ &- \left(\beta\left(1+\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{6}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) \\ &- \left(\beta^2\left(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha^2}{4}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) \\ &- \left(\beta^3\left(\frac{1}{6}+\frac{\alpha}{6}\right)\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) \\ &- \left(\frac{\beta^4}{24}\right)(L_{-1}(c)+L_0(c)\zeta+L_1(c)\zeta^2+L_2(c)\zeta^3+L_3(c)\zeta^4) \end{aligned}$$

satisfies the root condition $|\zeta_i| \le 1$.

Then the stability polynomial for this method is

$$S(\alpha,\beta:\zeta) = \zeta^{n+1} - \left(1+1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}\right)\zeta^n - \left(\beta+\frac{\beta^2}{2}+\frac{\beta^3}{6}+\frac{\beta^4}{24}+\frac{\alpha\beta^2}{2}+\frac{\alpha^2\beta}{2}+\frac{\alpha\beta^3}{6}+\frac{\alpha^3\beta}{6}+\frac{\alpha^2\beta^2}{4}+\alpha\beta\right)\zeta$$

The corresponding stability region is given in Fig. 1.

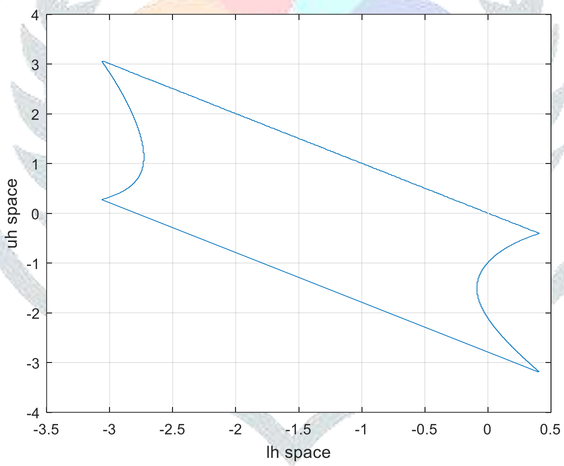

Fig. 1. Stability region of new one-step method

V. NUMERICAL EXAMPLES

Problem 1: (Stiff linear system with multiple delays)

$$y_1'(t) = -\frac{1}{2}y_1(t) - \frac{1}{2}y_2(t-1) + f_1(t),$$

$$y_2'(t) = -y_2(t) - \tfrac{1}{2}y_1\left(t-\tfrac{1}{2}\right) + f_2(t), \qquad 0 \le t \le 1$$

with initial conditions,

$$y_1(t) = e^{-t/2}, \qquad \tfrac{-1}{2} \le t \le 0,$$

$$y_2(t) = e^{-t}, \qquad -1 \le t \le 0$$

and $f_1(t) = \frac{1}{2}e^{-(t-1)}$, $f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}$

The exact solution is,

$$y_1(t) = e^{-t/2}, \qquad y_2(t) = e^{-t}$$

By taking the step-size h = 0.01 in this method, the approximate value and the absolute error are given in Tables 1-2. Their graph is presented in Fig. 2-3.

Table 1 Solution of y_1 in Example 1

t	y_1	Exact	Absolute error
0.1	0.9512296234	0.9512294245	1.988753e-07
0.2	0.9048377958	0.9048374180	3.777464e-07
0.3	0.8607085130	0.8607079764	5.365609e-07
0.4	0.8187314275	0.8187307531	6.744531e-07
0.5	0.7788015730	0.7788007831	7.899672e-07
0.6	0.7408191019	0.7408182207	8.812456e-07
0.7	0.7046890359	0.7046880897	9.461861e-07
0.8	0.6703210286	0.6703200460	9.825730e-07
0.9	0.6376291398	0.6376281516	9.881841e-07
1.0	0.6065316206	0.6065306597	9.608773e-07

Table 2 Solution of y_2 in Example 1

t	y_2	Exact	Absolute error
0.1	0.9048384551	0.9048374180	1.037073e-06
0.2	0.8187326031	0.8187307531	1.850012e-06
0.3	0.7408206835	0.7408182207	2.462844e-06
0.4	0.6703229384	0.6703200460	2.892396e-06
0.5	0.6065338108	0.6065306597	3.151105e-06
0.6	0.5488148852	0.5488116361	3.249069e-06
0.7	0.4965884992	0.4965853038	3.195370e-06
0.8	0.4493319631	0.4493289641	2.998959e-06
0.9	0.4065723290	0.4065696597	2.669257e-06
1.0	0.3790854026	0.3679074412	1.120596e-05

Problem 2: (Constant delay)

$$y'(t) = y(t-1), \qquad t \geq 0$$

with initial condition,

$$y(t) = e^{t}, \qquad t \leq 0$$

and the exact solution is,

$$y(t) = 1 + \frac{(-1+e^{t})}{e}, \quad t \geq 0$$

By taking the step-size h = 0.01 in this method, the approximate value and the absolute error are given in Table 3. Their graph is presented in Fig. 4 -5.

Table 3 Numerical Results of Example 2

t	y	Exact	Absolute error
0.1	1.0386895755	1.0386902186	6.430670e-07
0.2	1.0814481715	1.0814495229	1.351448e-06
0.3	1.1287037378	1.1287058626	2.124846e-06
0.4	1.1809292388	1.1809321949	2.956128e-06
0.5	1.2386473899	1.2386512185	3.828630e-06
0.6	1.3024358921	1.3024406049	4.712761e-06
0.7	1.3729332178	1.3729387795	5.561752e-06
0.8	1.4508450055	1.4508513119	6.306397e-06
0.9	1.5369511283	1.5369579769	6.848559e-06
1.0	1.6321135056	1.6321205588	7.053204e-06

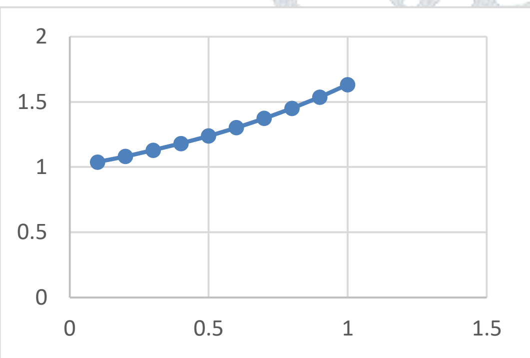

Fig. 4 Numerical solution of y

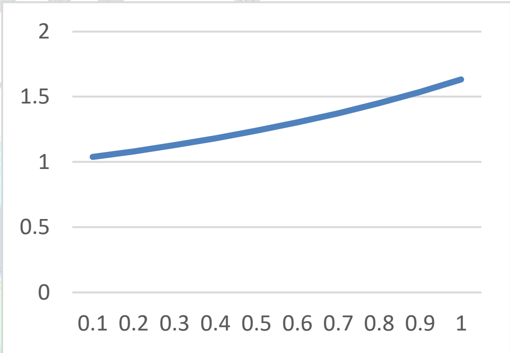

Fig. 5 Exact solution of y

Problem 3: (Time-dependent delay)

$$y'(t) = \frac{t-1}{t} y(\ln(t) - 1)y(t), \qquad t \geq 1$$

with initial condition,

$$y(t) = 1, \qquad t \leq 1$$

and the exact solution is,

$$y(t) = \exp(t - \ln(t) - 1), \qquad t \geq 1$$

By taking the step-size h = 0.01 in this method, the approximate value and the absolute error are given in Table 4. Their graph is presented in Fig. 6 -7.

Table 4 Numerical results of Example 3

t	Y	Exact	Absolute error
1.1	1.0046623512	1.0047008346	3.848338e-05
1.2	1.0177962004	1.0178356318	3.943139e-05

1.3	1.0383120404	1.0383529289	4.088848e-05
1.4	1.0655465006	1.0655890697	4.256916e-05
1.5	1.0991032513	1.0991475138	4.426253e-05

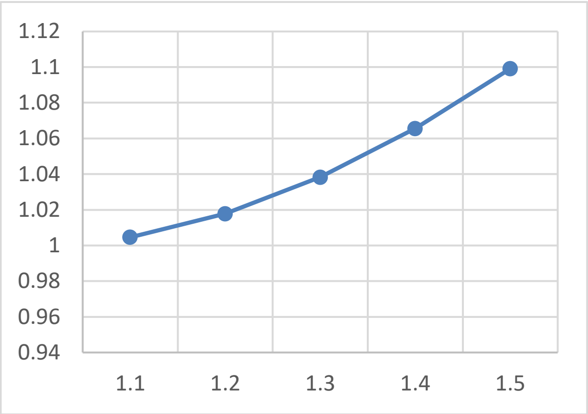

Fig. 6 Numerical solution of y

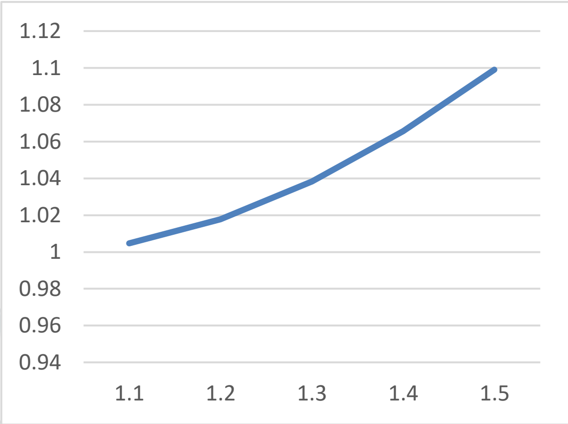

Fig. 7 Exact solution of y

Problem 4: (State-dependent delay)

$$y'(t) = \cos(t)y(\mathrm{y(t)} - 2), \qquad t \geq 0$$

with initial condition,

$$y(t) = 1, \qquad t \leq 0$$

and the exact solution is,

$$y(t) = \sin(t) + 1, \qquad 0 \leq t \leq 1$$

By taking the step-size h = 0.01 in this method, the approximate value and the absolute error are given in Table 5. Their graph is presented in Fig. 7-8.

Table 5 Numerical results of Example 4

t	y	Exact	Absolute error
0.1	1.0998350817	1.0998334166	1.665101e-06
0.2	1.1986726385	1.1986693308	3.307686e-06
0.3	1.2955251047	1.2955202067	4.898041e-06
0.4	1.3894247418	1.3894183423	6.399451e-06
0.5	1.4794333077	1.4794255386	7.769094e-06

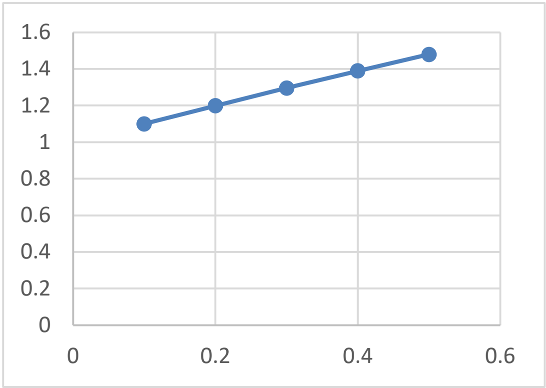

Fig. 7 Numerical solution of y

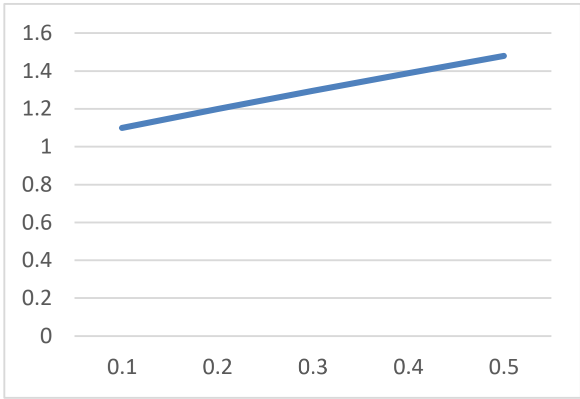

Fig. 8 Exact solution of y

VI CONCLUSION

In this paper, a new one-step method has been developed to solve the Delay differential equations (DDEs) by means of interpolating function which consists of rational functions. The delay argument is approximated using Lagrange interpolation. The stability polynomial of this method and the corresponding stability region have been determined. Numerical examples of stiff and non-stiff DDEs with constant delay, time dependent delay and state dependent delays have been considered to demonstrate the efficiency of this method. The numerical results reveals that this new method is suitable to solve DDEs. This new one-step method is computationally efficient, robust and easy to implement.

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