DIFFERENT APPROACHES TO GET AN INTEGER TRIPLES IN ARITHMETIC PROGRESSION

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ABSTRACT. We search for three non-zero distinct integer α,β,γ, such that the triples $(\alpha^2, (k^2+1)\gamma^3, \beta^2)$represent an Arithmetic progression. ,

KEYWORDS. Arithmetic Progression, Gaussian integer,

INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. In this communication we have two sections. In section 1 we have 4 cases which have a new approach to get an integer triplet in A.P using Pythagorean. Here we search for three non-zero distinct integer α,β,γ, such that the triples $(\alpha^2, (k^2+1)\gamma^3, \beta^2)$ forms an A.P

In Section 2 we have 3 cases using Gaussian integer to get an integer triplet in A.P. Here we search for three non-zero distinct α ,β,γ, such that the triples $(\alpha^2, (k^2+1)\gamma^3, \beta^2)$ forms an A.P .Where α,β,γ^3 in Gaussian integer.

Method of Analysis

Section 1

Let us choosing α, β, γ are any 3 three distinct non zero integer such that $(\alpha^2, (k^2+1)\gamma^3, \beta^2)$ forms an A.P (here k non zero integer)

By definition of A.P

$$2a-d = \alpha^2 \quad (1)$$

$$2a = (k^2+1)\gamma^3 \quad (2)$$

$$2a+d = \beta^2 \quad (3)$$

Eliminating a, d between (1) (2), (3)

$$\alpha^2 + \beta^2 = 2(k^2+1)\gamma^3 \quad (4)$$

Approach : 1

Let us considering $\alpha = (k^2+1)^2 x$ (5)

$$\beta = (k^2+1)^2 y \quad (6)$$

$$\gamma = (k^2+1)z \quad (7)$$

Substitute (5), (6),(7) in (4)

$$x^2 + y^2 = 2z^3 \quad (8)$$

Considering $x = r+s$ (9)

$$y = r - s \quad (10)$$

Substitute (9) (10) in (8)

$$r^2 + s^2 = z^3 \quad (11)$$

Case 1

Let us Choosing $z = p^2$ then $z^3 = (p^3)^2$

Let $Q = p^3$ then we get $z = Q^{2/3}$ (12)

Hence we get $r^2 + s^2 = Q^2$ (13)

By Pythagorean the solution of equation (13) is

$$r = a^2 - b^2 \quad (14)$$

$$s = 2ab \quad (15)$$

$$Q = a^2+b^2 \quad (16)$$

Substitute (14), (15), (16) in (9), (10), (12)

$$x = a^2-b^2 +2ab \quad (17)$$

$$y = a^2-b^2 - 2ab \quad (18)$$

$$z= (a^2+b^2)^{2/3} \quad (19)$$

Substitute (17), (18), (19) in (5), (6) (7)

$$\alpha = (k^2+1)^2 (a^2-b^2+2ab) \quad (20)$$

$$\beta = (k^2+1)^2 (a^2-b^2-2ab) \quad (21)$$

$$\gamma = (k^2+1) (a^2+b^2)^{2/3} \quad (22)$$

Substitute (20), (21), (22) in (1), (2), (3)

We get $2a\text{-}d = (k^2+1)^4 (a^2-b^2+2ab)^2$ (23)

$$2a = (k^2+1)^4 (a^2+b^2)^2 \quad (24)$$

$$2a+d = (k^2+1)^4 (a^2-b^2- 2ab)^2 \quad (25)$$

From (23), (24), (25) we get triplet forms A.P.

Example

	k=1, a =2, b=3	k=3, a=1, b=2	k=2 a=1 b=2
2a-d	784	10^4	2200
2a	2704	$25 \text{ x}10^4$	62500
2a+d	4624	$49 \text{ x x}10^4$	12,2500

Case 2

The solution of (11) be

$$r=m^3+mn^2 \quad (26)$$

$$s=m^2n+n^3 \quad (27)$$

$$z=m^2+n^2 \quad (28)$$

Subsitute (26),(27),(28) in (9) ,(10)

$x=(m^2+n^2)(m+n)$

$y=(m^2+n^2)(m-n)$

From (5),(6),(7)

$\alpha=(k^2+1)^2(m^2+n^2)(m+n)$ (29)

$\beta=(k^2+1)^2(m^2+n^2)(m-n)$ (30)

$\gamma=(k^2+1)(m^2+n^2)$ (31)

From (29),(30),(31)

$2a-d=(k^2+1)^4(m^2+n^2)^2(m+n)^2$ (32)

$2a=(k^2+1)^4(m^2+n^2)^3$ (33)

$2a+d=(k^2+1)^4(m^2+n^2)^2(m-n)^2$ (34)

From (32),(33),(34) we get triplet form in A.P

Example

	k=2,m=3,n=1	k=1, m=2, n=3	k=4, m=3, b=2
2a-d	1000,000	67600	352,876,225
2a	625000	35152	183,495,637
2a+d	2,50,000	2704	14,115,049

Case 3

Considering the linear transformation

$r=m^3-3mn^2$ (35)

$s=3m^2n-n^3$ (36)

$z=m^2+n^2$ (37)

Substitute (35),(36),(37) in (9) ,(10)

$x=(m-n)(m^2+4mn+n^2)$

$y=(m+n)(m^2-4mn+n^2)$

From (5),(6),(7)

$\alpha=(k^2+1)^2(m-n)(m^2+4mn+n^2)$ (38)

$\beta=(k^2+1)^2(m+n)(m^2-4mn+n^2)$ (39)

$\gamma=(k^2+1)(m^2+n^2)$ (40)

Substitute (38), (39),(40) in (1),(2),(3) we get

$2a-d=(k^2+1)^4(m-n)^2(m^2+4mn+n^2)^2$ (41)

$2a=(k^2+1)^4(m^2+n^2)^3$ (42)

$2a+d=(k^2+1)^4(m+n)^2(m^2-4mn+n^2)^2$ (43)

From (41),(42),(43) we get triplet form A.P

Example

	k=1, m =2, n=3	k=2, m=3, n=1	k=3, m=2, n=1
2a-d	21904	1210000	1690000
2a	35152	625000	125000
2a+d	48400	40000	810000

Case 4

Let us choosing $x=pz$ (44)

$y=qz$ (45)

$z=\frac{p^2+q^2}{2}$ (46)

put $p=2a$, $q=2b$ (47)

Substitute (44),(45),(46) in (47) we get

$x=4a(a^2+b^2)$ (48)

$y=4b(a^2+b^2)$ (49)

$z=2(a^2+b^2)$ (50)

Substitute (48),(49),(50) in (5),(6),(7)

$\alpha = (k^2+1)^2 4a(a^2+b^2)$ (51)

$\beta = (k^2+1)^2 4b(a^2+b^2)$ (52)

$\gamma = (k^2+1)2(a^2+b^2)$ (53)

Substitute (51), (52), (53) in (1), (2), (3)

$2a-d = (k^2+1)^4 16a^2(a^2+b^2)^2$ (54)

$2a = (k^2+1)^4 8(a^2+b^2)^3$ (55)

$2a+d = (k^2+1)^4 16b^2(a^2+b^2)^2$ (56)

From (54),(55),(56) we get triplet form of A.P

Example

	k=1, a=2, b=3	k=2, a=1, b=3	k=3, a=2, b=1
2a-d	173056	10000000	16000000
2a	281216	5000000	10000000
2a+d	389376	9000000	4000,000

Section 2

Case 1

Choosing a linear transformation as Gaussian integer

$r = m + in$ (57)

$s = (m+n) - in$ (58)

$Q = h - in$ (59)

Substitute (57), (58), (59) in (13)

$(2m+h)^2 = 2n^2+h^2$ (60)

Sub case (i)

Let $(2m+h) = 2a^2+b^2$ (61)

$n = 2ab$ (62)

$h = 2a^2-b^2$ (63)

using (61), (62), (63) substitute the values of m, n, h in (57), (58), (59), we get

$r = b^2+i2ab$ (64)

$s = 2a^2 – i2ab$ (65)

$Q = (2a^2-b^2) – i2ab$ (66)

From (9) ,(10) ,(12)

$x = 2a^2 + b^2$ (67)

$y = (b^2-2a^2) + i4ab$ (68)

$z = ((2a^2-b^2) –i2ab)^{2/3}$ (69)

From (5),(6),(7)

$\alpha = (k^2+1)^2 (2a^2+b^2)$ (70)

$\beta = (k^2+1)^2 ((b^2-2a^2) + i4ab)$ (71)

$\gamma = (k^2+1) ((2a^2-b^2)-i2ab)^{2/3}$ (72)

From (1),(2),(3)

$2a-d = (k^2+1)^4 (2a^2+b^2)^2$ (73)

$2a+d = (k^2+1)^4 ((b^2-2a^2) + i4ab)^2$ (74)

$2a = (k^2+1)^4 ((2a^2-b^2) –i2ab)^2$ (75)

From (73), (74), (75) we get triplet form of A.P.

Sub Case (ii)

Let $(2m+h) = a^2+2ab^2$ (76)

$n = 2ab$ (77)

$h = a^2-2b^2$ (78)

Using (76), (77), (78) substitute the values of m, n, h in (57), (58), (59)

$r = 2ab^2 + i2ab$ (79)

$s = a^2-i2ab$ (80)

$Q = (a^2-2b^2) – i2ab$ (81)

From (9), (10), (12)

$x = a^2+2b^2$ (82)

$y = (2b^2-a^2) + i4ab$ (83)

$z = ((a^2-2b^2) – i2ab)^{2/3}$ (84)

From (5),(6),(7)

$\alpha = (k^2+1)^2 (a^2+2b^2)$ (85)

$\beta = (k^2+1)^2 ((2b^2-a^2)+i4ab)$ (86)

$\gamma = (k^2+1) ((a^2-2b^2)-12ab)^{2/3}$ (87)

From (1),(2),(3)

$2a-d = (k^2+1)^4 (a^2+2b^2)^2$ (88)

$2a+d = (k^2+1)^4 ((2b^2-a^2) + i4ab)^2$ (89)

$2a = (k^2+1)^4 ((a^2-2b^2) – i2ab)^2$ (90)

From (88), (89), (90) we get triplet form in A.P

Example

	k=1, a=2, b=3	k=2, a=1, b=3	k=1,b=3, a=2,
2a-d	2^6 x 121	5^4 x 361	10^4x81
2a	2^6 (13+84i)	5^4 (253+204i)	10^4 (33+56i)
2a+d	2^6 (-95+168i)	5^4(145+408i)	10^4(-48+56i)

Case 2

Considering the linear transformation

$r = a-ib$ (91)

$s = a+ic$ (92)

$Q = a+id$ (93)

Substitute (91), (92), (93) in (13) and simplify we get

$a^2 = 2bc$ (94)

$d = c-b$ (95)

Consider $2b = u+v$, (96)

$c = u-v$ (97)

Substitute (96), (97) in (94)

$u^2 = a^2 + v^2$ (98)

which can be solved as

$u = f^2 + g^2$ (99)

$a = f^2 – g^2$ (100)

$v = 2fg$ (101)

we get $b = \frac{f^2+g^2+2fg}{2}$

$c = f^2+g^2-2fg$

$d = \frac{f^2+g^2-6fg}{2}$

Put f = 2F g = 2G

$a = 4F^2 – 4G^2$ (101)

$b = 2F^2 + 2G^2 + 4FG$ (102)

$c = 4F^2 + 4G^2 – 8FG$ (103)

$d = 2F^2 + 2G^2 – 12FG$ (104)

Substitute (101), (102), (103) in (9), (10), (12) we get

$x = 2a +i(c-b)$ (105)

$y = i(-b-c)$ (106)

$z =(a+id)^{2/3}$ (107)

subsitute(101), (102), (103), (104) in (105), (106), (107) and doing some simplification we get

$2a-d = (k^2+1)^4 ((8F^2-8G^2) + i(2F^2+2G^2-12FG))^2$ (108)

$2a = (k^2+1)^4 ((4F^2-4G^2) + i(2F^2+2G^2-12FG))^2$ (109)

$2a+d = (k^2+1)^4 (-1) (-6F^2 – 6G^2 + 4FG)^2$ (110)

From (108), (109), (110), we get triplet form in A.P

Example

	k=1, F = 2, G=3	k=2, F=1, G=3	k = 3, F=1, G=2
2a-d	2^6 (-129+ 920i)	5^4 x 16^2 x (15+8i)	10^4 x 2^2 x (95 +168i)
2a	2^6 (-429 + 460i)	5^4 x 16^2 x (3+4i)	10^4 x 2^2 x (-13+84i)
2a+d	2^6(-729)	5^4 x 16^2 (-9)	10^4x2^2 x(-121)

Case 3

Considering the linear transformation as

$r = a+it$ (111)

$s =b+it$ (112)

$Q = c+it$ (113)

(12) reduces to $t^2 = -2ab$ (114)

$c = a+b$ (115)

By taking $ab = -2\alpha^2$ and considering (114), doing some simplification

we get $(a-b)^2 = c^2 +8 \alpha^2$ (116)

By Pythagorean the solution of (116) be

$a-b = 8p^2 + q^2$

$c = 8p^2 – q^2$

$d = 2pq$

Hence $a = 8p^2$

$b = q^2$

$t = 4pq$

$c = 8p^2-q^2$

From (111),(112),(113)

$r = 8p^2 + i4pq$

$s = -q^2 + i4pq$

$Q = (8p^2-q^2) + i4pq$

From (9),(10),(12)

$x = (8p^2-q^2) + i8pq$

$y = 8p^2 + q^2$

$z = ((8p^2-q^2) + i4pq)^{2/3}$

From (5),(6),(7)

$\alpha = (k^2+1)^2 ((8p^2-q^2)+i8pq)$ (117)

$\beta = (k^2+1)^2 (8p^2 + q^2)$ (118)

$\gamma = (k^2+1) ((8p^2-q^2) + i4pq)^{2/3}$ (119)

Substitute (117),(118),(119) in (1),(2),(3) we get

$2a-d = (k^2+1)^4 ((8p^2-q^2) + i8pq)^2$ (120)

$2a = (k^2+1)^4 ((8p^2-q^2) + i4pq)^2$ (121)

$2a+d = (k^2+1)^4 ((8p^2+q^2)$ (122)

From (120), (121), (122) we get triplet form in A.P

	k=1, p = 2, q=3	k=2, p-1, q=3	k = 3, p = 1, q = 2
2a-d	2^4 (-1775+2208i)	5^4 x [-575 - 48i]	10^4 x 4^2 (-15+8i)
2a	2^4 (-47 + 1104i)	5^4 x (-143-24i)	10^4 x 4^2 (-3+4i)
2a+d	1681	5^4 x (289)	10^4 x 4^2 (9)

CONCLUSION

In this session we have obtained infinitely many triples in A.P using the solution of Pythagorean and Gaussian .To conclude ,one may search for triples with certain properties obtained from other choices of Diophantine equation.

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