COEFFICIENT, DISTORTION AND GROWTH INEQUALITIES FOR CERTAIN IMMINENT CONVEX FUNCTIONS

ABSTRACT. In the present examination, certain subclasses of close – to-curved capacities are explored. Specifically, we acquire a gauge for the Fekete-Szego utilitarian for capacities having a place with the class, contortion, development gauges and covering theorems.

Keywords: Coefficient, Distortion, Growth Inequalities, Convex Functions.

1. INTRODUCTION

Let D :={z∈C : |z| < 1} be the open unit circle in the unpredictable plane C. Let A be the class of systematic capacities characterized on D and standardized by standardized by the conditions $f(0)=0$ and $f(0)=1$. Give S a chance to be the subclass of A comprising of univalent capacities. Sakaguchi [7] presented a class of capacities called starlike capacities as for symmetric pits; they are the capacities concerning symmetric focuses; they are the capacities $f \in A$ fulfilling the condition.

$$\mathrm{Rc}\,\frac{f'(z)}{f(z)-f(z)} > 0$$

These capacities are close – to curved capacities. This can be effectively observed by demonstrating that the capacity $(f(z) - f(-z))/2$ is a starlike capacity in D. Inspired by the class of starlike capacities as for symmetric focuses, Gao and Zhou [2] examined a class K_s of near raised univalent capacity. A capacity $f \in K_s$, in the event that it fulfill the accompanying imbalance

$$\mathrm{Re}\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < 0 \quad (z\in D)$$

For some capacity $g \in S^*(1/2)$. The thought here is to supplant the normal of (1-z), 0 $f(z)$ and by the relating item – g(z) g(- z) and the factor z is incorporated to standardize the articulation so that $-z^2 f(z)/(g(z)\, g(-z))$ takes the esteem 1 at z = 0. To make the capacities univalent, it is additionally expected that g is star like of request ½ with the goal that the some ongoing takes a shot at the issue [11, 9, 10, 12]. Instead of requiring the amount $-z^2 f'(z) / (g(z)\, g(-z)$ to lie in the right-half plane, we can think about more broad districts. This should be possible by means of subordination between diagnostic capacities.

Let f and g be analytic in D. Then f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ (z∈ D), if there is an analytic function w(z), with w(0) = 0 and |w(z) | < 1, such that $f(z) = g(w(z))$. In specific, if g is univalent in D, then f is secondary to g, if $f(0) = g(0)$ and $f(D) \subseteq g(D)$. With relations to reduction, a common class $K_s(\psi)$ is presented in the subsequent classification.

$$-\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) \prec \psi(z) \qquad (z\in D)$$

for some function $g \in S^*(1/2)$

This class was presented by Wang, Gao and Yuan [11]. A unique subclass $K_8(\gamma)$: = $K_8(\psi)$: = where $\psi(z)$:= (1 +1-2ϒ)z) (1 – z), 0 ≤ ϒ < 1, was as of late researched by Kowalczyk and Les-Bomba [5]. They demonstrated the sharp bending and development gauges for capacities in $K_8(\gamma)$ and some adequate conditions as far as the coefficient for capacity to be in this class $K_8(\gamma)$.

In the present examination, we acquire a sharp gauge for the Fekete-Szego practical for capacities having a place with the class $K_8(\gamma)$. Moreover, we additionally examine the relating issue for the converse capacities having a place with the class $K_8(\gamma)$. Additionally bending, development appraises and covering hypothesis are determined. Some association with prior works are additionally shown.

2. FEKETE-SZGO INEQULITY

In this section, we expect that the capacity ψ(z) is a univalent systematic capacity with positive genuine part that maps the unit plate D onto a starlike district which symmetric as for genuine pivot and is standardized by ψ (0) = 1 and ψ ́(0)> 0. In such case, the capacity has an extension of the frame $\psi(z) = 1 + B_1 z + B_1 z^2 + \ldots, B_1 > 0$.

Theorem 2.1 (Fekete –Szego Inequality). For a function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ belonging to the class $K_8(\psi)$, the following sharp estimate holds:

$|a_3 - \mu_2{}^2| \leq 1/3 + \max(B_1/3, |B_2/3 - \mu B_1{}^2/4|)$ (μ ∈ C)

Proof. Since the function $f \in K_8(\psi)$, there is a normalized analytic function $g \in S^*(1/2)$ such that

$$-\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) \prec \psi(z)$$

By utilizing he meaning of subordination between scientific capacity, we discover a capacity w(z) expository in D, standardized by w(0) = 0 fulfilling |w(z) < 1 and

$$(2.1) \qquad -\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) = \psi'(z)$$

By writing $w(z) = w_1 z + w_2 z^2 + \ldots$, we see that

$$(2.2) \qquad \psi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 z + B_1 w_1^2)z^2 + \ldots$$

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Also by writing $g(z) = g(z) = z + g_2 z^2 + g_3 z^3 + \ldots$, a calculation shows that

$$-\left(\frac{g(z)g(-z)}{z}\right) = 1 + (2g_3 - g_2^2)z^3 + \ldots$$

and therefore

$$-\left(\frac{z}{g(z)g(-z)}\right) = 1 - (2g_3 - g_2^2)z^3 + \ldots$$

Using this and the Taylor's expansion for $f'(z)$, we get

$$(2.3) \qquad -\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) = 1 + a_2 z + (3a^3 - 2g_3 + g_2^2)z^2 + \ldots$$

Using (2.1), (2.2) and (2.3), we see that

$2a_2 = B_1w_1$

$3a_3 = 2g_3 - g_2^2z^3 + B_1w_2 + B_2 - w_1^2z^3$.

This shows that

$a_3 - \mu a_2^2 = (2/3)(g_3 - g_2^2/2) + (B_2/B_1 - 3\mu B_1/4) w_1^2)$.

By using the following estimate (|4, inequality 7, p,10|)

$$|w_2 - tw_1^2| \le \max \{1; |t|\} (t \in C)$$

For an analytic function w with w(0) =0 and |w(z) < 1 which is sharp for the functions $w(z) = z^2$ or w(z) = z, the desired result follows upon using the estimate that $|(g_3 - g_2^2/2| \le 1/2$ for analytic function $g(z) = z + g_2z^2 + g_3z^3 + \dots$ which is starlike of order ½.

Define the function f_0 by

$f_0(z) = \int_0^z \frac{\psi(w)}{1-w^2} dw$

The function clearly belongs to the class $K_8(\psi)$ with g(z) = z/(1-z). Since

We have

$$\frac{\psi(w)}{1-w^2} = 1 + B_1w + (B_2+1)w^2 + \dots$$

We have

$$f_0(z) = z + (B_1/2)z^2 + (1/3 + B_2/3)z^3 + \dots$$

Similarly, define f_1

$$f_1(z) = \int_0^z \frac{\psi(w)}{1-w^2} dw$$

Then

$$f_1(z) = z + (B_1/3 + 1/3) z^3 + \dots$$

The functions f_0 and f_1 show that the results are sharp.

Remark 2.1 . By setting μ=0 in theorem 2.1, we get the sharp estimate for the third coefficient of functions in $K_8(\psi)$:

$$|a_3| \le 1/3 + (B_1/3) \max (1, |B_2| /B_1).$$

While the limiting case μ→∞ gives the sharp estimate $|a_2| \le B_1/2$. In the special case where ψ(z) = (1 +z) (1 – z), the results reduce to the corresponding one in

Thought u et al [12] have given an estimate of $|a_n|$ for all n, their result is not sharp in general. For n = 2, 3, out results provide sharp bounds.

It is known that every univalent function f has an inverse f^{-1}, defined by

$f^{-1}(f(z) = z, \quad z \in D$

and

$f(f^{-1}(w) = w, \quad \left(|w| < r_0(f); r_0(f) \ge \frac{1}{4}\right)$

Corollary 2.1 , Let $f \in K_8(\psi)$. Then the coefficients d_2 and d_3 of the inverse function $f^{-1}(w) = w + d_2w^2 + d_3w^3 + \dots$ satisfy the inequality.

$|d_3 - \mu d_2^2| \le 1/3 + \max (B_1/3, |B_2/3 - (2-\mu) B_1^2/4| \quad (\mu \in C)$

Proof : A calculation shows that the inverse function f^{-1} has the following Taylor's series expansion.

$$f(f^{-1}(w) = w + a_2w^2 + 2 \quad a_2^2 - a_3)w^3 + \dots$$

From this expansion, it follows that $d_2 = a_2$ and $d_3 = 2a_2^2 - a_3$ and hence

$$|d_3 - \mu d_2^2| = |a3 - (2-\mu)\ a_2^2|.$$

Our result follows at once from this identity and Theorem

3. DISTORTION AND GROWTH THEOREMS

The second coefficient of univalent capacity assumes an essential part in the hypothesis of univalent capacity; for instance, this prompts the bending and development gauges for univalent capacities and also the turn hypothesis. In the following hypothesis, we determine the contortion and development gauges for the capacities in the class $K_8(\psi)$. Specifically, on the off chance that we let r → 1− in the development gauge, it gives the bound |a2| ≤ B1/2 for the second coefficient of capacities in $K_8(\psi)$.

Theorem 3.1. Let ' be an analytic univalent functions with positive real part and

$$\phi(-r) = \min_{|z|=r<1} |\phi(z)|, \quad =\phi(r) = \max_{|z|=r<1} |\phi(z)|,$$

If $f \in K_8(\psi)$,the following sharp inequalities holds:

$$\frac{\psi(-r)}{1+r^2} \le |\ '(z)| \le \frac{\psi(r)}{1-r^2} (|z| = r < 1)$$

$$\int_0^r \frac{\psi(-t)}{1+t^2} dt \le |\ (z)| \le \int_0^r \frac{\psi(t)}{1-t^2} dt (|z| = r < 1)$$

Proof. Since the function $f \in K_8(\psi)$, there is a normalized analytic function $g \in S^*(1/2)$ such that

$$= \frac{z^2\ '(z)}{g(z)g(-z)} \prec \psi(z)$$

Define the function G : D → C by the equation

$$G(z) := - = \frac{g(z)g(-z)}{z}$$

Then it is clear that G is odd starlike function in D and therefore

$$\frac{r}{1+r^2} \le |G(z)| \le \frac{r}{1-r^2} (|z| = r < 1)$$

Using the definition of subordination between analytic function, and the equation, we see that there is an analytic function w(z) with |w(z)| ≤ |z| such that

$$= \frac{z f'(z)}{G(z)} = \psi(w(z))$$

or $zf'(z) = G(z)\psi(w(z))$. Since w(D) ⊂ D, we have, by maximum principle for harmonic functions,

$\sqrt{1+z^2}$

The other inequality for |f′(z)| is similar. Since the function f is univalent, the inequality for |f(z)| follows from the corresponding inequalities for |f′(z)| by Privalov's Theorem

To prove the sharpness of our results, we consider the functions

$$f_0(z) = \int_0^z \frac{\psi(w)}{1-w^2} dw, \quad f_1(z) = \int_0^z \frac{\psi(w)}{1+w^2} dw$$

Define the function g0 and g1 by $g_0(z) = z/(1-z)$ and $g_1(z) = z/\sqrt{1+z^2}$. These functions are clearly starlike functions of order 1/2. Also a calculations shows that

Subsequently the capacity f_0 fulfills the subordination (1.1) with g0 while the capacity f_1 fulfills it with g1; accordingly, these capacities have a place with the class $K_s(\psi)$. Unmistakably the upper evaluations for |f′(z)| and |f(z)| are sharp for the capacity f0 given in (3.2) while the lower gauges are sharp for f1 given in (3.2).

Remark 3.1. We note that Xu et al. [12] also obtained a similar estimates and our results differ from their in the hypothesis. Also we have shown that the results are sharp.

Our hypothesis is same as the one assumed by Ma and Minda [6].

Remark 3.2. For the choice $\psi(z) = (1+z)/(1-z)$, our result reduces to [2, Theorem 3, p. 126] while, for the choice $\psi(z) = (1 + (1 - 2)z)/(1 - z)$, it reduces to following estimates (obtained in [5, Theorem 4, p. 1151]) for $f \in K_s(\gamma)$:

where $|z| = r < 1$. Also our result improves the corresponding results in [11].

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Remark 3.3. Let $k := \lim_{r\to 1^-} \int_0^r \psi(-t)/(1 + t^2)dt$. Then the disk $\{w \in C : |w| \le k\} \subseteq f(D)$ for every $f \in K_s(\psi)$.

4. A SUBORDINATION THEOREM

It is outstanding [8] that f is starlike if $(1 - t)f(z) \prec f(z)$ for $t \in (0, \delta)$, where is a positive genuine number; likewise the capacity is starlike as for symmetric focuses if $(1-t)f(z)+ tf(-z) \prec f(z)$. In the accompanying hypothesis, we stretch out these outcomes to the class Ks. The verification of our outcome depends on the accompanying adaptation of a lemma of Stankiewicz.

Lemma 4.1. Let $F(z, t)$ be analytic in D for each $t \in (0, \delta)$, $F(z, 0) = f(z)$, $f \in S$ and $F(0, t) = 0$ for each $t \in (0, \delta)$. Suppose that $F(z, t) \prec f(z)$ and that

exists for some $\rho > 0$. If F is analytic and Re(F(z)) 6= 0, then

Theorem 4.1. Let $f \in S$ and $g \in S^*(1/2)$. Let $\delta > 0$ and $f(z) + tg(z)g(-z)/z \prec f(z)$, $t \in (0, \delta)$. Then $f \in K_s$.

Proof. Define the function F by $F(z, t) = f(z) + tg(z)g(-z)/z$. At that point $F(z, t)$ is explanatory for each settled t and $F(z, 0) = f(z)$ and by our presumption, $f \in S$.

Moreover, the capacity F is explanatory in D (obviously, one needs to reclassify the capacity F at $z = 0$ where it has removable peculiarity.) Since all the speculation of Lemma 4.1 are fulfilled, we have

Since a capacity p(z) has negative genuine part if and just if its complementary 1/p(z) has negative genuine part, we have

Thus, $f \in$ Ks.

REFERENCES

[1] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[2] C. Gao and S. Zhou, On a class of analytic functions related to the starlike functions, Kyungpook Math. J. 45 (2005), no. 1, 123–130.

[3] A. W. Goodman, Univalent Functions, Vol. 1, Mariner, Tampa, 1983

[4] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 (1969), 8–12.

[5] J. Kowalczyk and E. Leś-Bomba, On a subclass of close-to-convex functions, App. Math. Lett., 23 (2010), no. 10, 1147–1151.

[6] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, MA.

[7] K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11 (1959), 72–75.

[8] J. Stankiewicz, Some remarks on functions starlike with respect to symmetric points, Ann. Univ. Mariae Curie-Sklodowska Sect. A 19 (1965), 53–59 (1970).

[9] Z.-G. Wang and D.-Z. Chen, On a subclass of close-to-convex functions, Hacet. J. Math. Stat. 38 (2009), no. 2, 95–101.

[10] Z.-G. Wang, C.-Y. Gao and S.-M. Yuan, On certain new subclass of close-to-convex functions, Mat. Vesnik 58 (2006), no. 3-4, 119–124.

[11] Z. Wang, C. Gao and S. Yuan, On certain subclass of close-to-convex functions, Acta Math. Acad. Paedagog. Nyhazi. (N.S.) 22 (2006), no. 2, 171–177 (electronic).

[12] Q.-H. Xu, H.M. Srivastava, Z. Li, A certain subclass of analytic and close-to-convex functions, App. Math. Lett., In Press.