AUGMENTED SUSTAINIBILITY ON COEFFICIENTS OF ALPHA OGARITHMICALLY CURVED FUNCTIONS

ABSTRACT: A scholarly exposition of some cutting edge frontiers which seeks to give an overview of the star like sharp bound convex functions relying on some of the great theorems and adepts to explain its essence.

For $\alpha \geq 0$, here M^α be group of α - logarithmically convex capacities. i.e. *f* analytic in $z \in D = \{z : |z| < 1\}$ satiating

$$\mathrm{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right]>0.$$

Some cutting edge frontiers for the initial coefficients of the inverse function f^{-1} of $f \in M^\alpha$ are given.

Keywords: Univalent functions, starlike, convex, α - logarithmically convex functions, inverse functions, coefficients.

1. Introduction. Here *S* be group of analytic normalised univalent functions *f*, demarcated in $z \in D = \{z : |z| < 1\}$ and given by

$$(1.1)\quad f(z)=z+\sum_{n=2}^{\infty}a_n z^n.$$

Signify by S^* the subclass of capapcities, starlike w.r.t the source and by *C* the subgroup of convex capacities.

Thus $f \in S^*$ if, and only if, for $z \in D$,

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0,$$

and $f \in C$ if, and only if,

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0.$$

For $\alpha \geq 0$, let M^α denote the class of functions satisfying

$$\mathrm{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\right]>0$$

for $z \in D$.

We arrive the conclusion that the class M^α is the power analogue of the so-called Ma- Minda functions M^α, which have been widely studied [8]. It was shown in [1] that for $\alpha \geq 0, M^\alpha \subset S^*$ and, together with other results, sharp upper bounds for $|a_2|$, $|a_3|$ and the Fekete-Szego functional where found.

Since functions in M^α are univalent in *D*, they possess an inverse function f^{-1}, defined on some set of *f(D)* with radius at least 1/4.

Suppose that on such a set, f^{-1} has Taylor series expansion

$$(1.2)\quad f^{-1}(\omega)=\omega+\sum_{n=2}^{\infty}\gamma_n\omega^n.$$

It is the main purpose of this paper to find some sharp upper bounds of $|\Upsilon_n|$ for n= 2,3 and 4 for $f \in M^\alpha$.

Let $h \in P$, the class of functions with positive real part in *D* and write

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n.$$

We shall make use the following well-known results [2,6] :

Lemma 1. If $p \in P$, then $|c_n| \leq 2$ for $n \geq 1$, and

$$\left|c_2-\frac{\mu}{2}c_1^2\right|\leq\max\{2,2|\mu-1|\}=\begin{cases}2 & 0\leq\mu\leq 2,\\ 2|\mu-1| & \text{elsewhere.}\end{cases}$$

Lemma 2. If $h \in P$ with coefficients c_n as above, then for some complex valued *x* with $|x| \leq 1$ and some complex valued ζ with $|\zeta| \leq 1$

$$2c_2=c_1^2+x(4-c_1^2)$$

$$4c_3=c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)\zeta$$

We prove the following :

2. Theorem. Let $f \in M^\alpha$ and f^{-1} be given by (1.2). Then the following sharp inequalities hold

$$|\gamma_2|\leq\frac{2}{1+\alpha},$$

$$|\gamma_3|\leq\frac{5+7\alpha}{(1+\alpha)^2(1+2\alpha)},$$

$$|\gamma_3|\leq\frac{5+7\alpha}{(1+\alpha)^2(1+2\alpha)},$$

$$|\gamma_4|\leq\frac{2(63+77\alpha+3\alpha^2+\alpha^3)}{9(1+\alpha)^3(1+3\alpha)}.$$

The inequalities for $|\gamma_2|$ and $|\gamma_3|$ are valid when $\alpha \geq 0$ and for $|\gamma_4|$ when $0 \leq \alpha < \approx 2.04$.

Proof, Write

$$(2.1)\quad \left(1+\frac{zf''(z)}{f'(z)}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}=h(z).$$

Since $f(f^{-1}(\omega)) = \omega$, then(1.1) and (1.2) give

$$(2.2)\quad \gamma_2=-a_2,$$

$$\gamma_3=-a_3+2a_2^2,$$

$$\gamma_4=-a_4+5a_2a_3-5a_2^3,$$

and equating coefficients in (2.1), we obtain

$$(2.3)\quad a_2=\frac{c_1}{1+\alpha},$$

$$a_3=\frac{(2+7\alpha-\alpha^2)c_1^2}{4(1+\alpha)^2(1+2\alpha)}+\frac{c_2}{2(1+2\alpha)},$$

$$a_4=\frac{(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)c_1^3}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}+\frac{(3+19\alpha-4\alpha^2)c_1c_2}{6(1+\alpha)(1+2\alpha)}+\frac{c_3}{3(1+3\alpha)}$$

Since $|c_1| \le 2$, it follows at once from (2.2) and (2.3) that $|\gamma_2| \le 2/(1+\alpha)$.

For γ_3, (2.2) and (2.3) give

$$(2.4)\quad |\gamma_3| = \left| \frac{c_2}{2(1+2\alpha)} - \frac{(6+9\alpha+\alpha^2)c_1^2}{4(1+\alpha)^2(1+2\alpha)} \right|.$$

Applying Lemma 1 with $\mu = \frac{6+9\alpha+\alpha^2}{2(1+\alpha)^2}$, so that $\mu \notin [0,2]$, we obtain

$$|\gamma_3| \le \frac{(5+7\alpha)}{(1+\alpha)^2(1+2\alpha)},$$

Which proves the second inequality in the Theorem.

For γ_4, we obtain from (2.2) and (2.3)

$$(2.5)\quad |\gamma_4| = \left| \frac{c_3}{3(1+3\alpha)} - \frac{(6+\alpha)c_1c_2}{3(1+\alpha)(1+3\alpha)} + \frac{(48+73\alpha+21\alpha^2+2\alpha^3)c_1^3}{18(1+\alpha)^3(1+3\alpha)} \right|.$$

We now use Lemma 2 to express c_3 and c_2 in terms of c_1 to obtain

$$|\gamma_4| = \left| \frac{(63+77\alpha+3\alpha^2+\alpha^3)c_1^3}{36(1+\alpha)^3(1+3\alpha)} - \frac{5c_1x(4-c_1^2)}{6(1+\alpha)(1+3\alpha)} - \frac{c_1x^2(4-c_1^2)}{12(1+3\alpha)} + \frac{(4-c_1^2)(1-|x|^2)\zeta}{6(1+3\alpha)} \right|.$$

Without loss is generally we may normalise the coefficient c_1 and assume that $c_1 = c$, where $0 \le c \le 2$. Then using the triangle inequality we obtain

$$|\gamma_4| \le \frac{(63+77\alpha+3\alpha^2+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)} - \frac{5c|x|(4-c^2)}{6(1+\alpha)(1+3\alpha)} + \frac{c|x|^2(4-c^2)}{12(1+3\alpha)} + \frac{(4-c^2)(1-|x|^2)}{6(1+3\alpha)} = \psi(c,|x|).$$

Assume now that ψ (c,|x|) has a critical point inside [0, 2]x[0, 1], then differentiating with respect to |x| and equating to zero implies $c = 2$, which is a contradiction.

Thus in order to find the maximum of ψ (c,|x|), we need only consider the end points of [0, 2J x [0, 1],

On $c = 0$,

$$\psi(0,|x|) = \frac{2(1-|x|^2)}{3(1+3\alpha)} \le \frac{2}{3(1+3\alpha)}.$$

On $c = 2$

$$\psi(2,|x|) = \frac{2(63+77\alpha+3\alpha^2+\alpha^3)}{9(1+\alpha)^3(1+3\alpha)}.$$

On |x| = 0,

$$\psi(c,0) = \frac{(63+77\alpha+3\alpha+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)} + \frac{(4-c^2)}{6(1+3\alpha)}.$$

This expression is minimum at p=0, and maximum at

$$p = \frac{4(1+\alpha)^3}{(63+77\alpha+3\alpha^2+\alpha^3)} \text{ for } \alpha \ge 0.$$

Substituting back gives

$$\psi(c,0) \le \frac{2(63+77\alpha+3\alpha+\alpha^3)}{9(1+\alpha)^3(1+3\alpha)}.$$

Finally on |x| = 1,

$$\psi(c,1) \le \frac{(63+77\alpha+3\alpha^2+\alpha^3)c^3}{36(1+\alpha)^3(1+3\alpha)} + \frac{(4-c^2)c}{12(1+3\alpha)} + \frac{5(4-c^2)}{6(1+\alpha)(1+3\alpha)}.$$

This expression increase with *2* on [0, 2] provided $0 \le \alpha \approx 2.04$, which again gives the inequality for $|\gamma_4|$.

Taking $c_1 = c_2 = c_3 = 2$ in (2.4) and (2.5) shows that the inequalities in the Theorem are sharp.

Remark 1. The investigation demonstrates that the disparities for |γ_2 | and |γ_3| are legitimate for α ≥0, and for |γ_4 | when 0 ≤ α ≈ 2.04, Finding the sharp imbalance for |γ_4 | for all α ≥0 remains an open inquiry.

Remark 2. At the point when α = 0, the imbalances acquired in the Theorem are predictable with those given by Lowner [7] for f ∈S, and when α =l, with those given in [5] for curved capacities.

Remark 3. For genuine μ and v, the useful J4(f) = |a_4+μa_2 a_3+va_3^2 | assumes a critical job in the hypothesis of univalent capacities. In [9], sharp limits for J4(f) for all genuine μ, and v were acquired for the Ma-Minda capacities Ma¬ . This paper finds the sharp destined for J4(f) for capacities in Mα for the situation μ =-5 and v= 5, (for the fourth opposite coefficient). Discovering sharp limits for JA (f) for all genuine μ and v when f ∈ Mα remains an open issue. Specifically acquiring the correct upper headed for JA(f) when μ = - l and v= 1/3 would give an answer for the issue of finding the sharp destined for the third coefficient of log(f(z)/z) for f ∈ Mα. Anyway the technique utilized in this paper does not seem to give the sharp bound for this situation.

Remark 4. At the point when 0 ≤ α ≤ 1, utilizing the disparities |cn| ≤ 2 for n= 1,2,3 in (2.3) above and a comparable articulation for α5 regarding c1,c2,c3 and c4, effectively sets up the accompanying, which are sharp when c1=c2 =c3=c4 =2.

$$|a_2| \le \frac{2}{1+\alpha},$$

$$|a_3| \le \frac{3(1+3\alpha)}{(1+\alpha)^2(1+2\alpha)},$$

$$|a_4| \le \frac{2(18+113\alpha+292\alpha^2+7\alpha^3+2\alpha^4)}{9(1+\alpha)^3(1+2\alpha)(1+3\alpha)},$$

$$|a_5| \le \frac{45+629\alpha+2908\alpha^2+2969\alpha^3+12061\alpha^4+812\alpha^5+196\alpha^6}{9(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)}.$$

Following indistinguishable technique from above gives sharp upper limits for |α_n | for n ≥ 6 for capacities in Mα which are processable, yet the estimations turn out to be progressively entangled.

Kulshrestha [3, 4], discovered sharp upper limits for the coefficients of capacities in Ma for n ≥ 2, yet since capacities in Mα include powers, taking care of the comparing issue for Mα for all n > 2 might be more troublesome.

We at long last note that finding the right request of development for the coefficients an as n→∞ for f ∈ Mα is additionally an open issue.

REFERENCES

1. M. Dams and D. K. Thomas, a-logarithmically convex functions, *Indian J. Pure. Appl. Math.*, 29, 10 (1998), 1049-1059.
2. P. L. Duren, In *Univalent Functions*, (Springer-Verlag, 1983), 114-115.
3. P. M. Kulshrestha, Coefficients for alpha-convex univalent functions, *Bull. Amer. Math. Soc.*, 80 No 2, (1974), 341-342.

4. P. M. Kulshrestha, Coefficient problem for alpha-convex functions, *Arch. Rat. Mech. Anal.*, ***54,*** (1974), 205-211.
5. R.J. Libera and E.J. Zlotkiewicz, Early Coefficients of the Inverse of a Regular Convex Function, *Proc. Amer. Math. Soc.*, 85 No. 2 (1982), 225-230.
6. R.J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P., *Proc. Amer. Math. Soc.*, ***87*** No. 2 (1983), 251257.
7. C. Lowner, Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises, I, *Math. Ann.* 89 (1923), 103-121.
8. W. C. Ma and **D.** Minda, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis, Tianjin* (Internat.Press, Cambridge, MA, 1994), 157-169.
9. D. V. Prokhorov and J. Szynal, Inverse coefficients for (a,P)-convex functions, *Ann.Univ. M. Curie-Skldowska*, 35 15 (1981), 125-143.