EXTENDED RESULTS ON (G,D)-NONBONDAGE NUMBER OF A GRAPH

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ABSTRACT:* *The (G,D)-Nonbondage number of a graph G denoted by $b_n\gamma_G(G)$ is defined as the maximum cardinality among all sets of edges $X \subseteq E(G)$ such that $\gamma_G(G-X)=\gamma_G(G)$. If $b_n\gamma_G(G)$ does not exist, we define $b_n\gamma_G(G)=0$. In this paper, we shall give some extended results on (G, D)-Nonbondage number of a graph.

Keywords: Domination, Geodomination, (G, D)-number, (G, D)-Bondage number and (G, D)-Nonbondage number.
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1. Introduction: Throughout this paper, we consider G as a finite undirected graph with no loops and multiple edges. The concept of domination in graphs was introduced by Ore [8]. Let G = (V, E) be any graph. A dominating set of a graph G is a set D of vertices of G such that every vertex in V–D is adjacent to atleast one vertex in D and the minimum cardinality among all dominating sets is called the domination number γ(G). The concept of geodominating (or geodetic) set was introduced by Buckley and Harary in [1] and Chartrand, Zhang and Harary in [2, 3, 4]. Let u, v ∈ V(G). A u-v geodesic is a u-v path of length d(u, v). A vertex x ∈ V(G) is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. A set S of vertices of G is a geodominating(or geodetic) set if every vertex of G lies on an x-y geodesic for some x, y in S. The minimum cardinality of a geodominating set is the geodomination(or geodetic) number of G and is denoted as g(G)[1, 2, 3, 4]. The concept of (G, D)-set was introduced by Palani and Nagarajan [9]. A (G, D)-set of G is a subset S of V(G) which is both a dominating and geodetic set of G. A (G, D)-set S of G is said to be a minimal (G, D)-set of G if no proper subset of S is a (G, D)-set of G. The minimum cardinality of all (G, D)-sets of G is called the (G, D)-number of G and it is denoted by γ_G(G). Any (G, D)-set of G of cardinality γ_G is called a γ_G-set of G [9, 10, 11].

Fink et al. [5] introduced the bondage number of a graph in 1990. The bondage number $b(G)$ of a graph G is the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$.

In [7], Kulli and Janakiram introduced the concept of the nonbondage number as follows: The nonbondage number $b_n(G)$ of G is the maximum cardinality of all sets of edges $X \subseteq E$ for which $\gamma(G-X)=\gamma(G)$ for an edge set X, then X is called the nonbondage set and the maximum one the maximum nonbondage set. If $b_n(G)$ does not exist, we define $b_n(G)=0$.

Let G = (V, E) be any graph and v ∈ V(G). The neighbourhood of v, written as N_G(v) or N(v) is defined by N(v) = {x ∈ V(G) : x is adjacent to v}. The *degree* of a vertex v in a graph G is defined to be the number of edges incident with v and is denoted by *deg* v. A vertex of degree zero is an isolated vertex and a vertex of degree one is a *pendant vertex* (or end vertex). Any vertex which is adjacent to a pendant vertex is called a support. A graph G is *complete* if every pair of distinct vertices of G are adjacent in G. A complete graph on p vertices denoted by K_p. A *clique* of a graph is a maximal complete subgraph. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is a *proper subgraph* of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. A *spanning subgraph* of G is a subgraph H of G with $V(G)=V(H)$. A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree*.

Definition 1.1:[6] The (G, D)-bondage number of a graph G denoted by bγ_G(G) is the least positive integer k such that there exists F ⊆ E(G) with $|F| = k$ and $\gamma_G(G-F) > \gamma_G(G)$. If no such k exists, it is defined to be ∞.

Remark 1.2:[6] (i) (G, D)-number is defined for connected graphs with at least two vertices [9]. So, let us assume that (G, D)-number of a disconnected graph is the sum of (G,D)-number of its components.

(ii) Also, assume that (G, D)-number of a graph with less than two vertices, that is, graph is a single vertex is 1.

Definition 1.3:[6] The (G, D)-nonbondage number of a graph G denoted by $b_n\gamma_G(G)$ is defined as the maximum cardinality among all sets of edges $X \subseteq E(G)$ such that $\gamma_G(G-X)=\gamma_G(G)$.

2. MAIN RESULTS

Proposition 2.1: Let D(r,s) be the double star obtained from K_2 by joining r pendant edges to one end and s pendant edges to the other end of K_2. Then, $b_n\gamma_G(D(r,s)) = r+s-2$.

Proof: Let u and v be the vertices of K_2. Then, r and s end vertices adjacent to u and v respectively, in D(r,s). Remove r-1 pendant edges incident with u and s-1 pendant edges incident with v. The value of γ_G does not change. Obviously, removal of more edges results in the increase of γ_G-value. Therefore, $b_n\gamma_G(D(r,s)) = r + s - 2$.

Theorem 2.2: Let T be a tree with l end vertices and k support vertices such that $l + k = p$. Let L and K denote the set of all end and support vertices of T respectively. If $\gamma_G(T) = l$ and each support vertex is adjacent to at least two end vertices, then $b_n\gamma_G(T) = k - 1 + \sum_{v\in K}(deg_T v - 2)$.

Proof: Let K be the set of all support vertices of T. Suppose S is a (G, D)-set of T. Clearly, no support vertex of T belongs to S. Since $|K|=k$, the number of edges between the support vertices is exactly $k - 1$.

Step 1: Remove the $k - 1$ edges between the support vertices from T

Let the new graph be T'. Clearly, $T' \cong G_1 \cup G_2 \cup \dots \cup G_k$, where each G_i is a star of order atleast 3.

Therefore, $\gamma_G(T') = \gamma_G[G_1 \cup G_2 \cup \dots \cup G_K]$

$$= \gamma_G(G_1) + \gamma_G(G_2) + \cdots + \gamma_G(G_k)$$
$$= l.$$

Step 2: From T', remove $deg_{T'}v - 2$ pendant edges incident with each support vertex v

Let the resultant graph be $T''$and in T'', let G_i' denote the resultant of G_i. Since each $G_i (1 \le i \le k)$ is a star, $\gamma_G(G_i) = \gamma_G(G_i')$. Therefore,

$$\gamma_G(T'') = \gamma_G(G_1') + \gamma_G(G_2') + \cdots + \gamma_G(G_k')$$
$$= \gamma_G(G_1) + \gamma_G(G_2) + \cdots + \gamma_G(G_k)$$
$$= l$$
$$= \gamma_G(T).$$

Clearly, T'' is obtained by removing $k - 1 + \sum_{v\in K}(\deg_T v - 2)$ edges from the graph T. Also, removal of atleast one more edge from T increases the γ_G-value. Therefore, $b_n\gamma_G(T) = k - 1 + \sum_{v\in K}(deg_T v - 2)$.

Theorem 2.3: Given a positive integer $k \ge 1$, there exists a graph G with $b_n\gamma_G(G) = k$.

Proof: Consider the graph G in Figure 2.1.

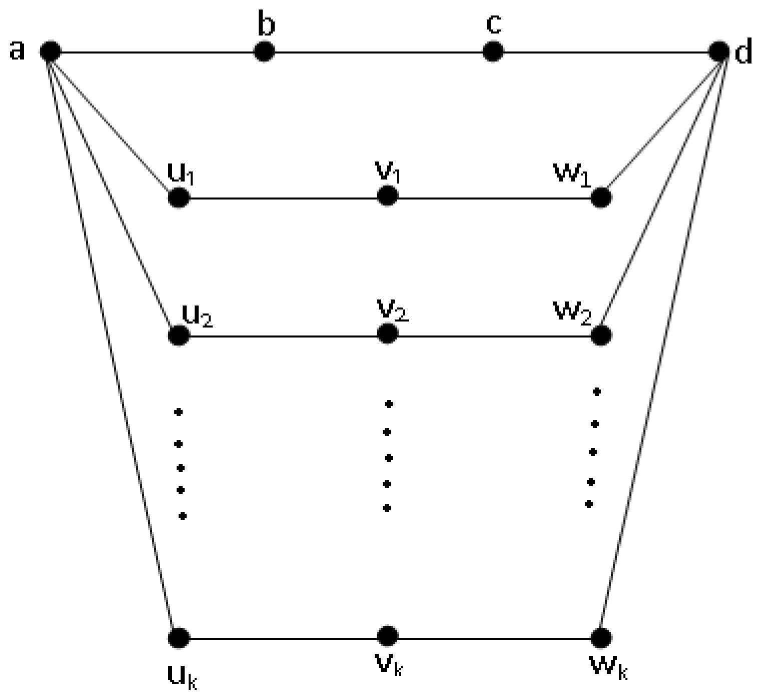

Figure (2.1)

Clearly, $\{a, d, v_i: 1 \le i \le k\}, \{a, d, u_i: 1 \le i \le k\}$ and $\{a, d, w_i: 1 \le i \le k\}$ are some minimum (G, D)-sets of G and so, $\gamma_G(G) = k + 2$.

Now, we remove the set of edges $X = \{au_i: 1 \le i \le k\}$ or $Y = \{dw_i: 1 \le i \le k\}$ from G. Then, $\{a, d, u_i: 1 \le i \le k\}$ and $\{a, d, w_i: 1 \le i \le k\}$ are the minimum (G,D)-sets of $G - X$ and $G - Y$ respectively. Thus, $\gamma_G(G - X) = \gamma_G(G - Y) = k + 2 = \gamma_G(G)$.

Also, removal of any $k + 1$ edges of G increases the γ_G-value of G.

Therefore, $b_n\gamma_G(G) = |X| = |Y| = k$.

Theorem 2.4: For any graph G, $b\gamma_G(G) \le b_n\gamma_G(G) + 1$ and the bound is sharp.

Proof: Let X be a $b_n\gamma_G$-set of G. Then, $X \cup \{e\}$ is a $b\gamma_G$-set of G.

So, $b\gamma_G(G) \le |X \cup \{e\}| = |X| + 1 = b_n\gamma_G(G) + 1$.

If $\cong C_4$, $b\gamma_G(G) = 2, b_n\gamma_G(G) = 1$ and so the bound is sharp.

Definition 2.5: An edge e of G is (G, D)-critical if $\gamma_G(G - e) > \gamma_G(G)$.

Definition 2.6: A graph G is called an edge (G, D)- critical (or edge γ_G – critical) graph if $\gamma_G(G - e) > \gamma_G(G)$ for every edge $e \in E(G)$.

Definition 2.7: An edge e of G is (G, D)-durable if $\gamma_G(G - e) = \gamma_G(G)$.

Definition 2.8: A graph G is called an edge (G, D)-durable (or edge γ_G-durable) graph if $\gamma_G(G - e) = \gamma_G(G)$ for every edge $e \in E(G)$.

Example 2.9: (i) In a star graph, every edge is γ_G-durable. So, star graph is a γ_G-durable graph.

(ii) P_2 is a γ_G - durable graph.
(iii) P_3 is a γ_G - critical graph.

Theorem 2.10: If for any edge e in G, there exists a $b_n\gamma_G$ - set containing e, then e is γ_G – durable.

Proof: Let $e \in E(G)$ such that there exists a $b_n\gamma_G$ - set S containing e. Then, $\gamma_G(G - S) = \gamma_G(G)$. Therefore, $\gamma_G(G - e) = \gamma_G(G)$ and so, e is γ_G - durable.

Corollary 2.11: If $b\gamma_G(G) = \infty$, then G is γ_G - durable.

Proof: Since $b\gamma_G(G) = \infty$, every edge of G belonging to $b_n\gamma_G$- set of G. By Theorem 2.10, every edge of G is γ_G - durable and hence, G is γ_G - durable.

Theorem 2.12: If $X \subseteq E(G)$ is a nonbondage (G, D)-set of G, then every edge in $G - X$ is γ_G-critical with respect to $G - X$.

Proof: Let X be a $b_n\gamma_G$-set of G. Then, $\gamma_G(G - X) = \gamma_G(G)$. If there exists an edge $e \in G - X$ such that e is not γ_G-critical, then $\gamma_G[(G - X) - \{e\}] = \gamma_G(G - X) = \gamma_G(G)$. So that,
$\gamma_G[G - (X \cup \{e\})] = \gamma_G(G)$. Which is a contradiction to the maximality of X.

Remark 2.13: Converse of the above theorem is not true. Consider the graph G given in Figure 2.2.

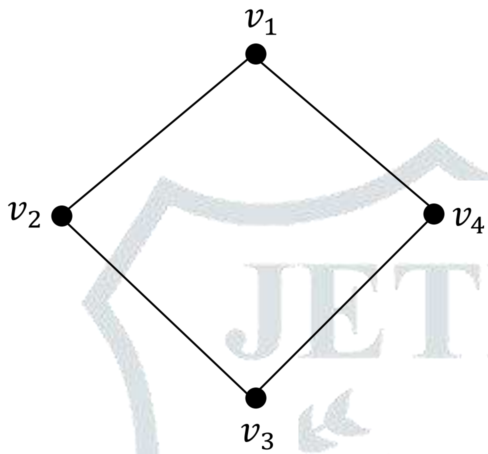

Figure (2.2)

Let $X = \{v_1v_2, v_1v_4\}$. Every edge in $G - X$ i[illegible] respect to $G - X$ but X is not a $b_n\gamma_G$-set of G.

Observation 2.14: Let G be a non-complete, non γ_G durable graph. If H is a spanning subgraph of G obtained by removing at least one γ_G-critical edge from G, then $\gamma_G(G) < \gamma_G(H)$.

Theorem 2.15: Let G be a graph without cliques of order greater than or equal to 3. Suppose H is a spanning subgraph of G. Then, $\gamma_G(G) \leq \gamma_G(H)$.

Proof: Suppose H is a spanning subgraph of G. Then, $V(G) = V(H)$ and $E(H) \subset E(G)$. Take $E' = E(G) - E(H)$.

Case 1: $|E'| = 0$

Then, $H = G$. Therefore, $\gamma_G(H) = \gamma_G(G)$.

Case 2: $|E'| = 1$

Let $E' = \{x\}$.

Subcase (2a): x is γ_G-critical

Then, $\gamma_G(G) < \gamma_G(H)$.

Subcase (2b): x is γ_G-durable

Then, $\gamma_G(G - x) = \gamma_G(G)$ and so, $\gamma_G(H) = \gamma_G(G)$.

Case 3: $|E'| > 1$

Subcase (3a): E' is a $b_n\gamma_G$-set

Then, $\gamma_G(G - E') = \gamma_G(G)$ and so, $\gamma_G(H) = \gamma_G(G)$.

Subcase (3b): E' is a $b\gamma_G$-set

Then, $\gamma_G(G - E') > \gamma_G(G)$ and so, $\gamma_G(H) > \gamma_G(G)$.

Subcase (3c): Every element of E' is γ_G-durable

Suppose $|E'| = |E| - 1$. Let $E' = \{x_1, x_2, \ldots, x_{q-1}\}$.

Then, $\gamma_G(G - x_1) = \gamma_G(G - x_2) = \cdots = \gamma_G(G - x_{q-1}) = \gamma_G(G)$. If we remove $q - 1$ edges from G, then $E(H) = 1$ and $V(H) = V(G)$. Therefore, $\gamma_G(H) = |V(H)| = |V(G)| \geq \gamma_G(G)$ and so, $\gamma_G(G) \leq \gamma_G(H)$.

Subcase (3d): Every element of E' is γ_G-critical

Suppose $|E'| = |E| - 1 = |\{x_1, x_2, \ldots, x_{q-1}\}|$ as before. Then, by Observation 2.14,
$\gamma_G(G - x_i) > \gamma_G(G)$ for all $i = 1, 2, \ldots, q - 1$ and so, $\gamma_G(G - E') > \gamma_G(G)$.
Thus, $\gamma_G(G) < \gamma_G(H)$.

Subcase (3e): $E' = W \cup Z$, W is the set of γ_G-durable edges and Z is the set of γ_G-critical edges

Since E' contains at least one γ_G-critical edge, by Observation 2.14, $\gamma_G(G) < \gamma_G(H)$.

From the above cases, $\gamma_G(G) \leq \gamma_G(H)$.

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