ERRORS IN NUMERICAL SOLUTIONS

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Abstract: In this paper contained the numerical solution of differential equations with finite differences, a variety of schemes are available for the discretization of the problem. In many situations, questions arise regarding the round-off and truncation errors involved in the numerical computations, as well as the stability and the convergence of the finite difference scheme. Here we presented a brief description of the physical significance of these terminologies and its different terms such as Round-off Errors, Truncation Error, Stability, and Convergence. Later, the third-order partial differential equations have been discussed since it can make a base of many mathematical models for the dynamics of the soil moisture and subsoil waters, spreading of acoustic waves in a weakly heterogeneous environment. Finally, fourth order convergence of the approximate solution is established.

Keywords: Round-off Errors, Truncation Error, Partial Differential Equations.

Errors involved in numerical solutions

Study of numerical solution of differential equations with finite difference solutions, a variety of structures are available for the discretization of the problem. In many cases, there are many questions arise regarding the truncation errors and round-off those involves in the numerical computations, as well as the stability and the convergence of the finite difference scheme. Here we have composed a brief description of the physical significance of these terminologies.

Round-off Errors

Computations are rarely made in exact arithmetic. This means that real numbers are represented in ”floating point” form and as a result, errors are caused due to the rounding-off of real numbers. Even though modern computers represent numbers to twelve or more places of decimals, in extreme cases such errors, called round-off errors, accumulate and become a main source of error.

Truncation Error

Infinite difference representation of derivatives with Taylor's series expansion, the higher order terms are neglected by truncating the series and the error caused due to such truncation is called the truncation error. For example, in forward difference of the first derivative to the order ΔX as follows

$$O(\Delta X) = -\frac{1}{2}(\Delta X) f^{''}(X_0) - \frac{1}{6}(\Delta X)^2 f^{'''}(X_0) + ...$$

Represents the truncation error and the lowest order term on the right hand side that has given the order of the method.

Stability

Numerical solution of differential equations with finite differences, errors are introduced at almost every stage of the calculations. The solution scheme is said to be stable if the error involved in numerical computations is not amplified without bounds.

Convergence

The numerical solution is said to be convergent if the numerical solution approaches the exact solution of the problem at the time and space steps tend to zero. We note that the conditions of stability and convergence are related to each other. The total error involved in finite difference calculations consists of the discretization error plus the round-off error. The discretization error increases with increasing the mesh size while the round-off error decreases with increasing mesh size. Therefore, the total error is expected to exhibit a minimum as the mesh size is decreased.

Method of Lines for Third Order Partial Differential Equations

The third-order partial differential equations make a base of numerous mathematical models for the flow of the dirt, moisture and subsoil waters, spreading of acoustic waves in a pitifully heterogeneous environment. Numerous physical wonders and mechanical situations have been formulated into boundary value problems with integral boundary conditions.

Later, many works have appeared such as Ashyralyev and Aggez, Ashyralyev and Tetikoglua, Pulkina, and Ashyralyev and Gercek. It ought to be noticed that there are so much work given to the existence of a solution for this sort of boundary value problems where explanatory equations, hyperbolic equations, and mixed-sort equations are considered. The proof of existence and uniqueness of solution has been contemplated by Latrous and Memou.

As of late, the reproducing portion space method (RKSM) assumes a vital part in numerical solutions of differential and integral equations. The main ideas of RKSM are based on the construction of reproducing piece space (RKS). The reproducing part function can retain all definite conditions. Then the numerical solution of definite problem is approximated by the reproducing portion function. Clearly constructing a reasonable reproducing portion space and viably calculating the reproducing part function expression become the way to apply RKSM. The method of lines is applied to the boundary-value problem for third order partial differential equation. Unequivocal expression and order of convergence for the estimated solution are obtained.

Be that as it may, because of the intricate three-point value conditions with an integral condition, the RKSM has not constructed reasonable RKS to manage the numerical solution. All the more precisely, the establishment of traditional RKS depends intensely on the two endpoints. Henceforth it cannot be reached out to three-point nonlocal boundary value problem which is based on intermediate point, especially with the integral boundary condition. In addition, to the best of the authors' information, the numerical approximations of the problem equation have not been considered before. Persuaded by all the works above, we describe a change of the RKSM to find the numerical solution. Another RKS is effectively established by a few strategies. Furthermore, other partial differential equations with multipoint boundary value conditions might be numerically solved using a comparable procedure.

Furthermore, we consider the boundary value problem for the third order differential equation in the domain $\Omega\{0<x<m, 0<y<n\}$:

$$\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2}=f(x,y) \quad (1)$$

$$w(x,0)=\phi_1(x), w(x,n)=\phi_2(x), \quad (2)$$

$$w(0,y)=h_1(y), w(m,y)=h_2(y) \quad (3)$$

$$\frac{\partial u}{\partial x}(0,y)=h_3(y) \quad (4)$$

Where $\phi_1(x),\phi_2(x),h_1(y),h_2(y),h_3(y)$ are sufficiently smooth functions.

The problems of type (1) -(4) arise in many mathematical and scientific applications. In this study, we construct first order accurate differential difference scheme for this problem and give error estimate for its solutions. The approach to the construction of the discrete problem and the error analysis for the approximate solution are similar.

Let the solution of the problem (1) -(4) have a bounded derivative $\frac{\partial^7 w}{\partial x \partial y^6}$ in the domain.

We divide the domain Ω into $t+1$ stripe by lines $y=y_l=l\,g(l=1,2,....t; g=\frac{n}{t+1})$. On these lines the problem of (1) -(4) we approximate by the following differential difference problem:

$$W_{l+1}'''+10W_l'''+W_{l-1}'''+\frac{12}{g^2}\left(W_{l+1}'-2W_l'+W_{l-1}'\right)=f_{l+1}(x)+10f_l(x)+f_{k-1}(x) \quad (5)$$

$$W_0(x)=\phi_1(x), W_{n+1}(x)=\phi_2(x) \quad (6)$$

$$W_l(0)=h_1(y_l), W_l(m)=h_2(y_l) \quad (7)$$

$$W_l(0)=h_3(y_l) \quad (8)$$

Let us rewrite the problem (5) -(8) in the form

$$AW'''-\frac{12}{g^2}MW'=F(x) \quad (9)$$

$$W(0)=h_1^{(0)}, W(m)=h_2^{(m)}, W'(0)=h_3^{(0)},$$

Where

$$W=(W_1(x),W_2(x),...,W_n(x)),$$

$$M=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

$$A=12I-M=\begin{bmatrix} 10 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 10 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 10 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 10 \end{bmatrix}$$

$$F(x)=(F_1(x),F_2(x),....,F_n(x)), F_1(x)=f(x,y_2)+10f(x,y_1)+f(x,0)-\phi_1'''(x)-\frac{12}{g^2}\phi_1'(x),$$

$$F_l(x)=f(x,y_{l+1})+10f(x,y_l)+f(x,y_{l-1}),l=2,3,....,t-1$$

$$F_l(x)=f(x,n)+10f(x,y_t)+f(x,y_{t-1})-\phi_1^{'''}(x)-\frac{12}{g^2}\phi_1^{'}(x),$$

$$h_1^{(0)}=(h_1(y_1),h_1(y_2),...,h_1(y_t)),h_2^{(m)}=(h_2(y_1),h_2(y_2),...,h_2(y_t)),h_3^{(0)}=(h_3(y_1),h_3(y_2),...,h_3(y_t)).$$

The matrix M can be diagnosed as

$M=B^{-1}$ Diagonal,

With

$$B=B^{-1}=(b_{ls})_{l,s=1}^{t}=\left((-1)^{l+s}\sqrt{\frac{2}{t+1}}\sin\frac{\pi ls}{t+1}\right)_{l,s=1}^{t},$$

$$\lambda_s=\frac{4}{g^2}\cos^2\left(\frac{\pi s}{2(t+1)}\right),s=1,2,...,t.$$

Multiplying Equation (9) on the left by B we have

$$(12-\lambda_s)\psi_s^{'''}-\frac{12}{g^2}\lambda_s\psi_s^{'}=k_s(x), \quad (10)$$

$$\psi_s(0)=\psi_{s0},\psi_s(m)=\psi_{sm} \quad (11)$$

$$\psi_s^{'}(0)=\psi_{s0}^{'},s=1,2,...,t, \quad (12)$$

Where

$$\psi_s(x)=\sum_{l=1}^{t}b_{sl}U_l(x),$$

$$k_s(x)=\sum_{l=1}^{t}b_{sl}F_l(x),$$

$$\psi_{s0}=\sum_{l=1}^{t}b_{sl}h_1(y_l),$$

$$\psi_{sm}=\sum_{l=1}^{t}b_{sl}h_2(y_l),$$

$$\psi_{s0}^{'}=\sum_{l=1}^{t}b_{sl}h_3(y_l),s=1,2,...,t.$$

The solution of (10) - (12) containing the third order ordinary differential equation with constant coefficients can be found explicitly

$$\psi_s(x)=-\int_0^m H_s(x,\zeta)\bar{k}_s(\zeta)d\zeta+\frac{\sinh\alpha_s m-\sinh\alpha_s(m-x)-\sinh\alpha_s x}{\alpha_s(\cosh\alpha_s m-1)} \quad (13)$$

$$\left[\psi_{s0}^{'}-\frac{1}{\sinh\alpha_s m}\int_0^m\sinh\alpha_s(m-\zeta)\bar{k}_s(\zeta)d\zeta\right]+\frac{\cosh\alpha_s x-1}{\cosh\alpha_s m-1}+\frac{\cosh\alpha_s m-\cosh\alpha_s m}{\cosh\alpha_s m-1}\psi_{s0}$$

Where

$$H_s(x,\zeta)=\begin{cases}\dfrac{\sinh\alpha_s x.\sinh\alpha_s(m-\zeta)}{\alpha_s\sinh\alpha_s m},x\le\zeta\\ \dfrac{\sinh\alpha_s(m-x).\sinh\alpha_s\zeta}{\alpha_s\sinh\alpha_s m},x\le\zeta\end{cases} \quad (14)$$

Therefore the solution of (5)-(8) can be expressed as

$$W_l(x)=\sum_{s=1}^{t}(-1)^{l+s}\sqrt{\frac{2}{t+1}}\sin\frac{\pi ls}{t+1} \quad (15)$$

$$\{-\int_0^m H_s(x,\zeta)\bar{k}_s(\zeta)d\zeta+\frac{\sinh\alpha_s m-\sinh\alpha_s(m-x)-\sinh\alpha_s x}{\alpha_s(\cosh\alpha_s m-1)}\left[\psi_{s0}^{'}-\frac{1}{\sinh\alpha_s m}\int_o^m \sinh\alpha_s(m-\zeta)\bar{k}_s(\zeta)d\zeta\right]$$
$$+\frac{\cosh\alpha_s m-\cosh\alpha_s x}{\cosh\alpha_s m-1}\psi_{s0}+\frac{\cosh\alpha_s x-1}{\cosh\alpha_s m-1}\psi_{sm}\} \qquad (16)$$

Where

$$\bar{k}_s(x)=\frac{1}{12-\lambda_s}\int_0^x k_s(n)dn$$

Now we investigate the error of the approximate solution. From the error $z_l(x)=w(x,y_l)-W_l(x)$ we have the following boundary value problem:

$$z_{l+1}^{'''}+10z_{l-1}^{'''}+\frac{12}{g^2}(z_{l+1}^{'}-2z_l^{'}+z_{l-1}^{'})=R_l(x), \qquad (17)$$

$$z_0(x)\equiv 0,\ z_{l+1}(x)\equiv 0,\ z_l(0)\equiv 0,\ z_l(m)\equiv 0,\ z_l^{'}(0)\equiv 0,\ l=1,2,...,t,$$

Where,

$$R_l(x)=-\frac{g^4}{20}.\frac{\partial^7 w}{\partial x\partial y^6}(x,\bar{y}_l),|R_l(x)|\le C_1 g^4, C_1=const, l=1,2,...,t, y_{l-1}<\bar{y}_l<y_{l+1}$$

$$(12-\lambda_s)\psi_s^{'''}-\frac{12}{g^2}\lambda_s\psi_s^{'}=k_s(x), \psi_s(0)=0, \psi_s(m)=0, \psi_s^{'}(0)=0$$

Or

$$\psi_s^{'''}-\alpha_s^2\psi_s=\bar{k}_s(x)+L_s, \psi_s(0)=0, \psi_s^{'}(0)=0, \psi_s(x)=\sum_{l=1}^t b_{sl}\chi_l(x), \bar{k}_s(x)=\int_0^x\left(\sum_{l=1}^t b_{sl}R_l(n)\right)dn$$

Next for,

$$L_s=\frac{\int_0^m \sinh\alpha_s(m-\zeta).\bar{k}_s(\zeta)d\zeta}{\int_0^m \sinh\alpha_s(m-\zeta)d\zeta},\left(\sinh\alpha_s(m-x)>0\right) \qquad (18)$$

By the mean value theorem we have

$$L_s=-\bar{k}_s(\delta), 0<\delta<m \qquad (19)$$

Then

$$\bar{k}_s(x)+L_s=\bar{k}_s(x)-\bar{k}_s(\delta)=\bar{k}_s^{'}(\zeta)(x-\delta)=k_s(\zeta)(x-\delta) \qquad (20)$$

Since $k_s(x)=\sum_{l=1}^n b_{sl}R_l(x)$, then it follows that

$$\left|\bar{k}_s(x)+L_s\right|\le|k_s(\zeta)|m\le\max_{0\le x\le m}|k_s(x)|m\le C_1 m\sqrt{2ng^{3.5}} \qquad (21)$$

Further, we note that $\int_0^m H(x,\zeta)d\zeta\le\frac{1}{\alpha_s^2}$ ad

$$|\psi_s(x)|=\left|\int_0^m H_s(x,\zeta)\left[\bar{k}_s(\zeta)+L_s\right]d\zeta\right|\le\frac{1}{\alpha_s^2}\max_{0\le x\le m}\left[\bar{k}_s(\zeta)+L_s\right]\le\frac{C_1 m\sqrt{2ng^{3.5}}}{\alpha_s^2(12-\lambda_s)} \qquad (22)$$

$$=\frac{C_1 m\sqrt{ng^{3.5}}}{24\sqrt{2}\cos^2\frac{\pi s}{2(t+1)}}, s=1,2,...,t.$$

Hence

$$\left|z_l(x)\right| \le \sum_{s=1}^{n}\left|b_{ls}\right|\left|\psi_s(x)\right| \le \sqrt{\frac{2}{t+1}}.\frac{C_1 m\sqrt{n}}{24\sqrt{2}}g^{5.5}\sum_{s=1}^{t}\frac{1}{\cos^2\frac{\pi s}{2(t+1)}} \qquad (23)$$

Using here the inequality $\sin x > \frac{2}{\pi}x\left(0 < x < \frac{\pi}{2}\right)$,and taking into account $\sum_{l=1}^{t}\frac{1}{l^2} < \sum_{l=1}^{\infty}\frac{1}{l^2} = \frac{\pi^2}{6}$

It follows that

$$\left|z_l(x)\right| \le \frac{C_1 m g^6}{24}(t+1)^2\sum_{s=1}^{t}\frac{1}{(t+1-s)} \le \frac{C_1\pi^2 mn^2}{144}g^4, \qquad (24)$$

I.e., fourth order convergence of the approximate solution is established.

CONCLUSION

The numerical displaying of genuine problems, for example, earth shake, traffic flow, flag handling, viscoelastic problems offers an ascent to partial differential equations (PDEs). Sometimes, it is difficult to compute the correct solutions of partial differential equations by utilizing explanatory strategies. This is the major weakness of this approach, to beat this issue, mathematicians are inspired to create numerical strategies for the arithmetic and logical solutions of PDEs. At that point we discussed about the numerical arrangement of differential equations with finite contrasts, an assortment of plans is accessible for the discretization of the issue. In many case questions emerge with respect to the round-off and truncation blunders required in the numerical calculations, and also the stability and the joining of the finite distinction conspire. Here we introduced a short depiction of the physical centrality of these wordings and its distinctive terms, for example, Round-off Errors, Truncation Error, Stability, and Convergence. Later the third-arrange partial differential equations has been talked about since it can make a base of numerous numerical models for the dynamics of the dirt dampness and subsoil waters, spreading of acoustic waves in a feebly heterogeneous condition.

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