MIN-MAX COPULA

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***Abstract:**Let (X_1,Y_1), $(X_2,Y_2),\ldots,(X_n,Y_n)$ be independent and identically distributed pairs of random variables having joint distribution function H and copula C. Let the marginal distribution functions of X_i and Y_i be F and G respectively. Then the copula, C_{M_n}, of componentwise maximum is already in the literature. In this paper we derive the copula, C_{m_n}, of componentwise minimum. Then the copulas, C_{1n} (C_{n1}), of the minimum (maximum) of the first (second) component and the maximum (minimum) of the other are obtained. The corresponding survival copulas are also obtained.*

***Keywords:** Copula, Min–max copula, Joint distribution function, Joint survival function.*

1. Introduction

While answering a question raised by M. Frechet about multivariate distributions and their one dimensional margins, Abe Sklar introduced the notion of copula. The word copula is of Latin origin, which means, a link or bound or tie. It was first used in a statistical or mathematical sense by Sklar (1959). For a given multivariate distribution we can uniquely identify the corresponding marginal distribution functions. But the converse is not unique. That is, for a pair of univariate distributions their joint distribution is not unique. Copulas are functions which connect joint distribution functions and their marginal distribution functions. i.e., copula evaluated at the margins gives a joint distribution function. This is the content of Sklar's theorem (Sklar 1959). We can also view this as a joint distribution function whose one dimensional margins are uniform (0, 1). It is also referred in literature as uniform representation and dependence function. It is a scale free measure of dependence and a starting point of construction of bivariate or multivariate distributions.

Nelson (1999), in his introductory text book on copulas describes the study of copulas and the role they play in probability, statistics and stochastic process. This book tells us that the study of copulas and their applications is in its infancy. However, during the last decade a few papers appeared in the statistics literature where multivariate extremes are modeled using the ideas of copulas. See for example Embrechts et al (2000), Mikosch (2006), Genest and Remillard (2006), Genest and Favre (2007).

In this paper we aim to obtain the copula corresponding to maximum of one component and minimum of other component for a sequence of independent and identically distributed bivariate random variables. The following section, Section 2, discuss the definition and basic concepts in copula theory which we require. Section 3 derives the copula of componentwise minimum. In Section 4, we derive the copula corresponding to the maximum of one component and minimum of other. Section 5 is a concluding section.

2. Preliminaries

In this section the definition of copula and some important theorems on copula which we require in the subsequent sections are discussed based on Nelson (1999).

Definition 2.1. *If I=[0, 1], a two-dimensional copula is a function C from $I \times I \rightarrow I$ with the following conditions:*

1. *For every u, v in I,* $C(u,0)=0=C(0,v)$ *(i.e., C is grounded) and* $C(u,1)=u$ *and* $C(1,v)=v$ *(i.e., C has margins).*
2. *For every* $u_1 \le u_2$ and $v_1 \le v_2$ *in I,* $C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\ge 0$ *(i.e., C is 2-increasing)*

For every copula C and every *u, v in I,* $W(u,v)\le C(u,v)\le M(u,v)$, where $W(u,v)=max\,(u+v-1,0)$ and $M(u,v)=min(u,v)$ are respectively the Frechet-Hoeffding lower and upper bounds of *C*, and these bounds are themselves copulas. Random variables with copula *M* are called co monotonic and those with copula *W* are counter monotonic. The third important copula is the product copula given by $C(u,v)=\prod(u,v)=uv,$ is the copula of independent random variables.

A celebrated theorem in the copula literature is the Sklar's theorem, which shed light on the relationship between multivariate distribution functions and their univariate margins. It is stated below.

Theorem 2.1 (Sklar's Theorem). Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x, y in $\overline{R}$,

$$H(x,y)=C(F(x),\,G(y)) \qquad (2.1)$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $RanF \times RanG$, where $RanF$ represents the range of F. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (2.1) is a joint distribution function with margins F and G.

In the above theorem, random variables are not mentioned. It is true for general distribution functions and hence also for distribution functions

of random variables. The following theorem describes the copula of strictly monotone transforms of X and Y, if C is the copula between X and Y.

Theorem 2.2. Let X and Y be continuous random variables with copula $C_{X,Y}(u,v)$. Let α and β be strictly monotone on $RanX$ and $RanY$ respectively.

1. If α and β are both strictly increasing, then $C_{\alpha(X),\beta(Y)}(u,v)=C_{X,Y}(u,v)$.
2. If α is strictly increasing and β strictly decreasing, then $C_{\alpha(X),\beta(Y)}(u,v)=u-C_{X,Y}(u,1-v)$.
3. If α is strictly decreasing and β strictly increasing, then $C_{\alpha(X),\beta(Y)}(u,v)=v-C_{X,Y}(1-u,v)$.
4. If α and β are both strictly decreasing, then $C_{\alpha(X),\beta(Y)}(u,v)=u+v-1+C_{X,Y}(1-u,1-v)$.

Survival copulas are copulas that join joint survival function to their one dimensional marginal survival functions. The joint survival function of a pair of random variables (X, Y) with joint distribution function $H(x,y)=C(F(x),G(y))$ is given by $\overline{H}(x,y)=\overline{F}(x)+\overline{G}(y)-1+C(1-\overline{F}(x),1-\overline{G}(y))$ where $\overline{F}(x)$ and $\overline{G}(y)$ are the marginal survival functions of X and Y respectively. Let $\hat{C}(u,v)=u+v-1=c(1-u,1-v)$ be a function defined from $I\times I \to I$. Then we have $\overline{H}(x,y)=\hat{C}(\overline{F}(x),\overline{G}(y))$ and $\hat{C}$ is called the survival copula of (X, Y).

The following theorem gives the copula of componentwise maximum of independent and identically distributed pairs of random variables with copula C.

Theorem 2.3. If C is a copula and n a positive integer, then the function C_{M_n} given by

$$C_{M_n}(u,v)=C^n\left(u^{1/n},v^{1/n}\right),\ for\ u,v\ in\ I$$

is a copula. Furthermore, if $(X_i,Y_i),\ i=1,2,\ldots,n$ are n independent and identically distributed pairs of random variables with copula C. Then C_{M_n} is the copula of $X_{M_n}=max\{X_i\}$ and $Y_{M_n}=max\{Y_i\}$.

3. Copula of Componentwise Minimum

Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be independent and identically distributed pairs of random variables having a common joint distribution function H and corresponding copula function C. Let F be the marginal distribution function of X_i and G be that of Y_i. Let $X_{m_n}=min\{X_1,X_2,\ldots,X_n\}$ and $Y_{m_n}=\min\{Y_1,Y_2,\ldots,Y_n\}$. Then the distribution function $F_{m_n}(x)$ and $G_{m_n}(y)$ of X_{m_n} and Y_{m_n} respectively are given by $F_{m_n}(x)=1-\left[1-F_X(x)\right]^n$ and $G_{m_n}(y)=1-\left[1-G_Y(y)\right]^n$ and the joint distribution function of the componentwise minimum X_{m_n} and Y_{m_n} is given by

$$H_{m_n}(x,y)=1-\overline{F}_{m_n}(x)-\overline{G}_{m_n}(y)+\overline{H}_{m_n}(x,y)$$

In this section we obtain the copula corresponding to componentwise minimum which is the content of the following theorem.

Theorem 3.1. If C is a copula and n a positive integer, then the function C_{m_n} given by

$$C_{m_n}(u,v) = u + v - 1 + \left[(1-u)^{\frac{1}{n}} + (1-v)^{\frac{1}{n}} - 1 + C\left(1-(1-u)^{\frac{1}{n}}, 1-(1-v)^{\frac{1}{n}}\right)\right]^n \quad (3.1)$$

is a copula. Furthermore, if $(X_i, Y_i), i = 1,2,...,n$ are n independent and identically distributed pairs of random variables with copula C, then C_{m_n} is the copula of $X_{m_n} = min\{X_i\}$ and $Y_{m_n} = min\{Y_i\}$.

Proof. We have,

$$\begin{aligned} X_{m_n} &= min\{X_1, X_2, ..., X_n\} \\ &= -max\{-X_1, -X_2, ..., -X_n\} \end{aligned}$$

and

$$\begin{aligned} Y_{m_n} &= min\{Y_1, Y_2, ..., Y_n\} \\ &= -max\{-Y_1, -Y_2, ..., -Y_n\} \end{aligned}$$

Hence, the proof follows from Theorem 2.2 and Theorem 2.3.

Remark 3.2. If X_i and Y_i are independent, so are X_{m_n} and Y_{m_n}.

$i.e., \prod_{m_n}(u,v) = \prod(u,v)$

Remark 3.3. If X_i and Y_i co-monotonic, so are X_{m_n} and Y_{m_n}.

$i.e., M_{m_n}(u,v) = M(u,v)$

Remark 3.3. If X_i and Y_i counter-monotonic, X_{m_n} and Y_{m_n} are not so.

$i.e., W_{m_n}(u,v) \neq W(u,v)$

Survival copulas of C_{M_n} and C_{m_n} can be obtained by the usual method and are respectively given by

$$\hat{C}_{M_n}(u,v) = u + v - 1 + C^m\left((1-u)^{1/n}, (1-v)^{1/n}\right) \quad (3.2)$$

$$\hat{C}_{m_n}(u,v) = \left[u^{1/n} + v^{1/n} - 1 + C\left(1-u^{1/n}, 1-v^{1/n}\right)\right]^n \quad (3.2)$$

The following table gives the copula and survival copula of componentwise maximum when X_i and Y_i have copula C.

$(X_i Y_i)$	$C(u,v)$
(X_{M_n}, Y_{M_n})	$C_{M_n}(u,v) = C^n\left(u^{1/n}, v^{1/n}\right)$ $C_{M_n}(u,v) = u + v - 1 + C^n\left((1-u)^{1/n}, (1-v)^{1/n}\right)$
(X_{m_n}, Y_{m_n})	$C_{m_n}(u,v) = u + v - 1 + \left[(1-u)^{1/n} + (1-v)^{1/n} - 1\right.$ $\left.+ C\left(1-(1-u)^{1/n}, 1-(1-v)^{1/n}\right)\right]^n$ $C_{m_n}(u,v) = \left[u^{1/n} + v^{1/n} - 1 + C\left(1-u^{1/n}, 1-v^{1/n}\right)\right]^n$

4. Min-Max Copula

In this section we obtain the copula corresponding to joint distribution of the minimum of one component and the maximum of other component of n independent and identically distributed pairs of random variables, whose components are coupled using the copula C. This is the content of the following theorem.

Theorem 4.1 If C is a copula and n, a positive integer, then the function C_{1n} and C_{n1} given by

$$C_{1n}(u \cdot v) = v - \left[v^{1/n} - C\left(1-(1-u)^{1/n}, v^{1/n}\right)\right]^n \quad (4.1)$$

$$C_{n1}(u \cdot v) = u - \left[u^{1/n} - C\left(u^{1/n}, 1-(1-v)^{1/n}\right)\right]^n \quad (4.2)$$

are copulas, called the min-max copulas. Furthermore, if (X_i, Y_i), I = 1, 2,…, n are independent and identically distributed pairs of random variables with copula C, then C_{1n} is the copula of $X_{m_n} = \min\{X_i\}$ and $Y_{M_n} = \max\{Y_i\}$ and C_{1n} is the copula of $X_{M_n} = \max\{X_i\}$ *and* $Y_{m_n} = \min\{Y_i\}$.

Proof. The proof follows in the same lines of that of Theorem 3.1. □

Now, from the relationship between copula and survival copula we have

$$\hat{C}_{1n}(u,v) = u - \left[(1-v)^{1/n} - C\left(1-u^{1/n},(1-v)^{1/n}\right)\right]^n \tag{4.3}$$

and

$$\hat{C}_{n1}(u,v) = v - \left[(1-u)^{1/n} - C\left((1-u)^{1/n}, 1-v^{1/n}\right)\right]^n . \tag{4.4}$$

5. Conclusion

We saw that from a given copula, C, we can construct many other copulas, and hence bivariate distributions, by copula transformation method (See, Nelson (1999)) and some of them are obtained. The following table provides the bivariate exponential distributions constructed using the above transformations of the copula $C(u,v) = \frac{uv}{u+v-uv}$ when the marginals are $F(x) = 1-e^{-x}$ and $G(y) = 1-e^{-y}$.

Structure of Copula and Corresponding Bivariate Exponential Distribution Functions
$C(u,v) = \frac{uv}{u+v-uv}$ $H(x,y) = \frac{(1-e^{-x})(1-e^{-y})}{(1-e^{-x})+(1-e^{-y})-(1-e^{-x})(1-e^{-y})}$
$C_{M_n}(u,v) = \frac{uv}{\left[u^{1/n}+v^{1/n}-u^{1/n}v^{1/n}\right]^n}$ $H(x,y) = \frac{(1-e^{-x})(1-e^{-y})}{\left[(1-e^{-x})^{1/n}+(1-e^{-y})^{1/n}-(1-e^{-x})^{1/n}(1-e^{-y})^{1/n}\right]^n}$
$\hat{C}_{M_n}(u,v) = u+v-1+\frac{(1-u)(1-v)}{\left[(1-u)^{1/n}+(1-v)^{1/n}-(1-u)^{1/n}(1-v)^{1/n}\right]^n}$ $H(x,y) = 1-e^{-x}-e^{-y}+\frac{e^{-x}e^{-y}}{\left[e^{-x/n}+e^{-y/n}-e^{-x/n}e^{-y/n}\right]^n}$
$C_{m_n}(u,v) = u+v-1+\frac{(1-u)(1-v)\left[2-(1-u)^{1/n}-(1-v)^{1/n}\right]^n}{[1-(1-u)^{1/n}(1-v)^{1/n}]^n}$ $H(x,y) = 1-e^{-x}-e^{-y}+\frac{e^{-x}e^{-y}\left(2-e^{-x/n}e^{-y/n}\right)^n}{\left[1-e^{-x/n}e^{-y/n}\right]^n}$
$\hat{C}_{m_n}(u,v) = \frac{uv\left(2-u^{1/n}-v^{1/n}\right)^n}{\left[1-u^{1/n}v^{1/n}\right]^n}$ $H(x,y) = \frac{(1-e^{-x})(1-e^{-y})\left[2-(1-e^{-x})^{1/n}-(1-e^{-y})^{1/n}\right]^n}{\left[1-(1-e^{-x})^{1/n}(1-e^{-y})^{1/n}\right]^n}$

Structure of Copula and Corresponding Bivariate Exponential Distribution Functions

$$C_{1n}(u,v)=v-\left[v^{1/n}-\frac{(1-(1-u)^{1/n})v^{1/n}}{1-(1-u)^{1/n}+v^{1/n}-(1-(1-u)^{1/n})v^{1/n}}\right]^n$$

$$H(x,y)=1-e^{-y}-\left[(1-e^{-y})^{1/n}-\frac{(1-e^{-x/n})(1-e^{-y})^{1/n}}{(1-e^{-x/n})+(1-e^{-y})^{1/n}-(1-e^{-x/n})(1-e^{-y})^{1/n}}\right]^n$$

$$C_{n1}(u,v)=u-\left[u^{1/n}-\frac{u^{1/n}(1-(1-v)^{1/n})}{u^{1/n}+1-(1-v)^{1/n}-u^{1/n}(1-(1-v)^{1/n})}\right]^n$$

$$H(x,y)=1-e^{-x}-\left[(1-e^{-x})^{1/n}-\frac{(1-e^{-x})^{1/n}(1-e^{-y/n})}{(1-e^{-x})^{1/n}+(1-e^{-y/n})-(1-e^{-x})^{1/n}(1-e^{-y/n})}\right]^n$$

$$\hat{C}_{1n}(u,v)=u-\left[(1-v)^{1/n}-\frac{(1-u^{1/n})(1-v)^{1/n}}{1-u^{1/n}+(1-v)^{1/n}-(1-u^{1/n})(1-v)^{1/n}}\right]^n$$

$$H(x,y)=(1-e^{-x})\left[e^{-y/n}-\frac{(1-(1-e^{-x})^{1/n})e^{-y/n}}{1-(1-e^{-x})^{1/n}+e^{-y/n}-(1-(1-e^{-x})^{1/n})e^{-y/n}}\right]^n$$

$$\hat{C}_{n1}(u,v)=v-\left[(1-u)^{1/n}-\frac{(1-u)^{1/n}(1-v^{1/n})}{(1-u)^{1/n}+1-v^{1/n}-(1-u)^{1/n}(1-v^{1/n})}\right]^n$$

$$H(x,y)=(1-e^{-y})\left[e^{-x/n}-\frac{e^{-x/n}(1-(1-e^{-y})^{1/n})}{e^{-x/n}+1-(1-e^{-y})^{1/n}-e^{-x/n}(1-(1-e^{-y})^{1/n})}\right]^n$$

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