Intuitionistic Fuzzy Modular Lattices: An Exploration

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Abstract: Intuitionistic fuzzy modular lattice is the generalization of fuzzy modular lattice. This paper first introduce the notions of intuitionistic fuzzy modular pairs from intuitionistic fuzzy lattices by using certain conditions and by these notions of intuitionistic fuzzy modular pairs developed intuitionistic fuzzy modular lattices with certain examples. The characterizations of intuitionistic fuzzy modular lattice were established by using intuitionistic fuzzy modular pairs.

Keywords: Intuitionistic fuzzy lattice, intuitionistic fuzzy modular pair, intuitionistic fuzzy modular lattice.

1. Introduction

Lattice has been used in Mathematics since 18th century. Modular lattice is one of the most important types of lattices. The theory of fuzzy sets proposed by Lotfi A Zadeh in 1965 has achieved a great success in various fields. With the research of fuzzy sets in 1986, K. Atanassov [1] presented intuitionistic fuzzy sets which are very effective to deal with vagueness. The concept of intuitionistic fuzzy sets is a generalization of fuzzy sets. Bustince and Burillo introduced the concept of intuitionistic fuzzy relations and investigated some of its properties and Yon and Kim introduced the notion of intuitionistic fuzzy sublattices, filters and ideals. N. Ajmal and K.V. Thomas [8] initiated such types of study in the year 1994. It was later independently established by N. Ajmal that the set of all fuzzy normal subgroups of a group constitute a sublattice of the lattice of all subgroups and is modular lattice. S.Nanda [7] proposed the notion of fuzzy lattice using the concept of fuzzy partial ordering. The concept of fuzzy lattice introduced by N.Ajmal, S.Nanda and Wilcox.L.R [7, 8, 10] explained modularity in the theory of lattices. In this paper new definitions were introduced for intuitionistic fuzzy modular lattice by using intuitionistic fuzzy modular pairs and characterization theorem for intuitionistic fuzzy modular lattice was proved.

2. Preliminaries

2.1. Definition: [K. Atanassov, 1986] : Let X be a nonempty set. An intuitionistic fuzzy set A is an object having the form: $A = \{x, \mu_A(x), \nu_A(x); x \in X\}$ where the function $\mu_A: X \to [0,1]$ defines the degree of membership and $\nu_A: X \to [0,1]$ defines the degree of non-membership of each element $x \in X$ to the set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$; for every $x \in X$. For convenience we can use $A = (\mu_A, \nu_A)$.

2.1.1 Definition: [KulHur, S.Y.Jang, Y.B.Jun, 2005] Let X and Y be two sets. An intuitionistic fuzzy relation R from X to Y is an intuitionistic fuzzy set of $X \times Y$ characterized by the membershipfunction μ_R in $X \times Y$ and non-membership function ν_R in $X \times Y$ that is, $R = \{(x, y), \mu_R(x, y), \nu_R(x, y): x \in X, y \in Y\}$ an intuitionistic fuzzy relation R from X to Y will be denoted by $R(X \times Y)$.

2.1.2 Definition: [KulHur, S.Y.Jang, Y.B.Jun, 2005] An intuitionistic fuzzy relation R is

1. Reflexive if for every $x \in X$; $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$

2. Antisymmetric if for every $(x, y) \in X \times X$;

$$\mu_R(x, y) > 0 \; and \; \mu_R(y, x) > 0 \; or \; \nu_R(x, y) < 1 \; and \; \nu_R(y, x) < 1 \Rightarrow x = y$$

3. Transitive if $R \circ R \subset R$, where $\circ$ is max-min and min-max composition, that is $\mu_R(x, z) \geq \max_y \; [min\{\mu_R(x, y), \mu_R(x, y)\}]$ and

$$\nu_R(x, z) \leq \min_y [max\{\nu_R(x, y), \nu_R(y, z)\}]$$

2.1.3 Definition: [K.V.Thomas, N.Ajmal] Let L be a lattice and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ be an intuitionistic fuzzy set of L. Then A is called an intuitionistic fuzzy sublattice of L if the following conditions are satisfied

1. $\mu_A(x \vee y) \geq min\{\mu_A(x), \nu_A(y)\}$
2. $\mu_A(x \wedge y) \geq min\{\mu_A(x), \nu_A(y)\}$
3. $\nu_A(x \vee y) \leq max\{\nu_A(x), \nu_A(y)\}$
4. $\nu_A(x \wedge y) \leq max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in L$.

The set of all intuitionistic fuzzy sublattice is denoted by IFL.

2.1.4 Definition: [K.V.Thomas, N.Ajmal] A fuzzy subset μ of L is called a fuzzy sublattice of L if

1. $\mu\,(x \vee y) \geq \; min\,\{(\mu(x), \mu\,(y)\}$
2. $\mu\,(x \wedge y) \geq \; min\,\{(\mu(x), \mu\,(y)\}, \; forall \; x, y \; \in L.$

2.1.5 Definition: [Wilcox.L.R**]** Let L be a fuzzy lattice and $\mu(a), \mu(b)$ in L. Thus $(\mu(a), \mu(b))$ is called a fuzzy modular pair if

$$\mu(c) \vee \mu(a \wedge b) \; = \mu\,(c \vee a) \wedge \mu(b), for \; all \; \mu(c) \; \leq \; \mu(b) \; in \; L.$$

That is $\mu(c) \vee [\mu(a) \wedge \mu(c \vee b)] = \mu(c \vee a) \; \wedge \mu(c \vee b), for \; all \; \mu(c) \; in \; L.$

3. Intuitionistic Fuzzy Modular Pairs in Intuitionistic Fuzzy Modular Lattice

3.1. Theorem: Let $\langle L, \mu_L, \nu_L \rangle$ is an IFL on L if and only if μ_L and $1 - \nu_L$ are fuzzy lattices on L.

Proof: Let $\langle L, \mu_L, \nu_L \rangle$ is an IFL on L.Then by the properties

(1) $\mu_L(x \vee y) \geq min\{\mu_L(x), \mu_L(y)\}$

(2) $\mu_L(x \wedge y) \geq min\{\mu_L(x), \mu_L(y)\}$

(3) $\nu_L(x \vee y) \leq max\{\nu_L(x), \nu_L(y)\}$

(4) $\nu_L(x \wedge y) \leq max\{\nu_L(x), \nu_L(y)\}$

By condition (1) and (2) μ_L is a fuzzy lattice.

condition (3) $gives$ $1 - \nu_L(x \vee y) \geq min\{1 - \nu_L(x), 1 - \nu_L(y)\}$

$$(1 - \nu_L)(x \vee y) \geq min\{(1 - \nu_L)(x), (1 - \nu_L)(y)\} \qquad (L_1)$$

Condition (4) $gives$ $1 - \nu_L(x \wedge y) \geq min\{1 - \nu_L(x), 1 - \nu_L(y)\}$

$$(1 - \nu_L)(x \wedge y) \geq min\{(1 - \nu_L)(x), (1 - \nu_L)(y)\} \qquad (L_2)$$

By (L_1) and (L_2) $1 - \nu_L$ is a fuzzy lattice.

That is if $\langle L, \mu_L, \nu_L \rangle$ is an intuitionistic fuzzy lattice.

Then μ_L and $1 - \nu_L$ are fuzzy lattice.

Conversely assume that μ_L and $1 - \nu_L$ are fuzzy lattice

Hence we have

(1) $\mu_L(x \vee y) \geq min\{\mu_L(x), \mu_L(y)\}$

(2) $\mu_L(x \wedge y) \geq min\{\mu_L(x), \mu_L(y)\}$

(3) $(1 - \nu_L)(x \vee y) \geq min\{(1 - \nu_L)(x), (1 - \nu_L)(y)\}$

(4) $(1-\nu_L)(x \wedge y) \geq min\{(1-\nu_L)(x), (1-\nu_L)(y)\}$

Then(3)$becomes$ $1-\nu_L(x \vee y) \geq min\{1-\nu_L(x), 1-\nu_L(y)\}$

$$\nu_L(x \vee y) \leq max\{\nu_L(x), \nu_L(y)\} \qquad (L_3)$$

(4)$becomes$ $1-\nu_L(x \wedge y) \geq min\{1-\nu_L(x), 1-\nu_L(y)\}$

$$\nu_L(x \wedge y) \leq max\{\nu_L(x), \nu_L(y)\} \qquad (L_4)$$

By (1), (2), $(L_3)and$ (L_4) $weget$ $\langle L, \mu_L, \nu_L\rangle$ is an IFL.

3.1.1 Definition: If $\langle \mu_L(a), \nu_L(a)\rangle$ and $\langle \mu_L(b), \nu_L(b)\rangle$ is an intuitionistic fuzzy modular pair $[IFMP]$ if $\langle \mu_L(a), \mu_L(b)\rangle$ and $\langle (1-\nu_L)(a), (1-\nu_L)(b)\rangle$ are fuzzy modular pair $[FMP]$.

3.1.2 Definition: An IFL is said to be intuitionistic fuzzy modular lattice if

$$\mu_L(a \vee b) \wedge \mu_L(a \vee c) = \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$$

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) = \nu_L(a) \vee [\nu_L(b) \wedge \nu_L(a \wedge c)]$$

The set of all intuitionistic fuzzy modular lattices is denoted by $IFML$.

3.1.3 Theorem: Let $\langle L, \mu_L, \nu_L\rangle$ is an $IFML$ and $\langle \mu_L(a), \nu_L(a)\rangle$ and $\langle \mu_L(b), \nu_L(b)\rangle$ is an $IFMP$. Then

1) $\mu_L(c) \vee \mu_L(a \wedge b) = \mu_L(c \vee a) \wedge \mu_L(b)$ $for\ all$ $\mu_L(c) \leq \mu_L(b)$

2) $\nu_L(c) \wedge \nu_L(a \wedge b) = \nu_L(c \vee a) \vee \nu_L(b)$ $for\ all$ $\nu_L(b) \leq \nu_L(c)$

Proof: Let $\langle L, \mu_L, \nu_L\rangle$ is an $IFML$ and $\langle \mu_L(a), \nu_L(a)\rangle$ and $\langle \mu_L(b), \nu_L(b)\rangle$ is an $IFMP$.

Then by definition $\langle \mu_L(a), \mu_L(b)\rangle$ and $\langle (1-\nu_L)(a), (1-\nu_L)(b)\rangle$ are FMP.

Therefore $\mu_L(c) \vee \mu_L(a \wedge b) = \mu_L(c \vee a) \wedge \mu_L(b)$ $for\ all$ $\mu_L(c) \leq \mu_L(b)$

$$(1-\nu_L)(c) \vee (1-\nu_L)(a \wedge b) = (1-\nu_L)(c \vee a) \wedge (1-\nu_L)(b)$$

$$for\ all\ (1-\nu_L)(c) \leq (1-\nu_L)(b)$$

That is $1-\nu_L(c) \vee 1-\nu_L(a \wedge b) = 1-\nu_L(c \vee a) \wedge 1-\nu_L(b)$

$$for\ all\ 1-\nu_L(c) \leq 1-\nu_L(b)$$

$\nu_L^c(c) \vee \nu_L^c(a \wedge b) = \nu_L^c(c \vee a) \wedge \nu_L^c(b)\ for\ all\ \nu_L(c) \geq \nu_L(b)$

$\left(\nu_L(c) \wedge \nu_L(a \wedge b)\right)^c = \left(\nu_L(c \vee a) \vee \nu_L(b)\right)^c\ for\ all\ \nu_L(c) \geq \nu_L(b)$

$\nu_L(c) \wedge \nu_L(a \wedge b) = \nu_L(c \vee a) \vee \nu_L(b)\ for\ all\ \nu_L(c) \geq \nu_L(b)$

3.2 Example:

Consider an IFL, N_5 is in the following figures.

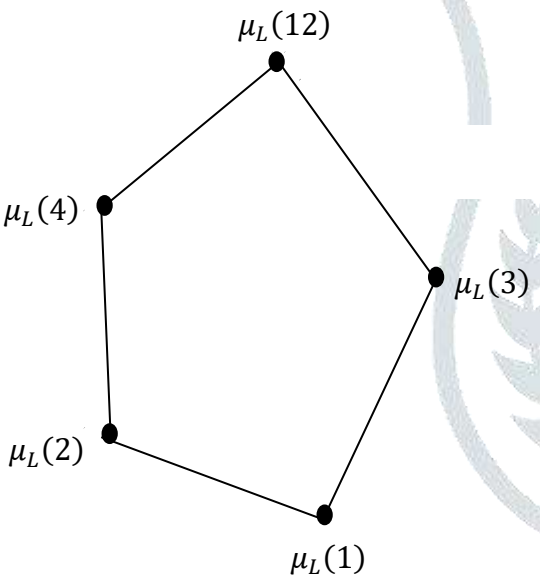

Figure 1

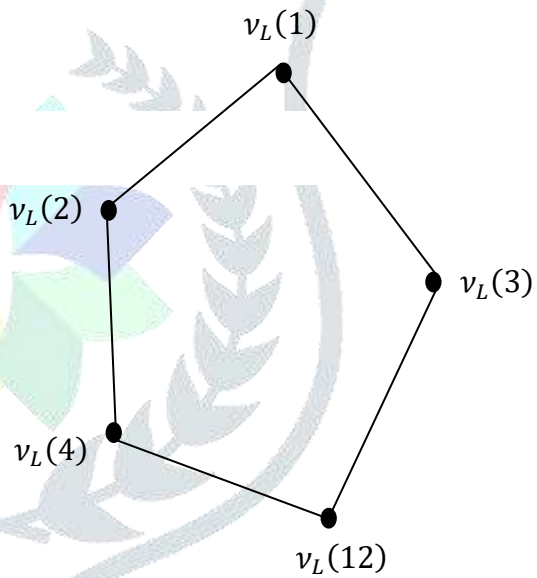

Figure 2

Consider $(a, b) = (4,12)$

Choose $\mu_L(c)$ such that $\mu_L(c) \leq \mu_L(b)$

$c = 1,2,3$

$\mu_L(c) \vee \mu_L(a \wedge b) = \mu_L(c \vee a) \wedge \mu_L(b)$ (1)

Put $c = 1$ in equation (1)

$\mu_L(1) \leq \mu_L(12)$

$\mu_L(1) \vee \mu_L(4 \wedge 12) = \mu_L(1 \vee 4) \wedge \mu_L(12)$

$\mu_L(1) \vee \mu_L(4) = \mu_L(4) \wedge \mu_L(12)$

$\mu_L(4) = \mu_L(4)$

Put $c = 2$ in equation (1)

$\mu_L(2) \leq \mu_L(12)$

$\mu_L(2) \vee \mu_L(4 \wedge 12) = \mu_L(2 \vee 4) \wedge \mu_L(12)$

$\mu_L(2) \vee \mu_L(4) = \mu_L(4) \wedge \mu_L(12)$

$\mu_L(4) = \mu_L(4)$

Put $c = 3$ in equation (1)

$\mu_L(3) \leq \mu_L(12)$

$\mu_L(3) \vee \mu_L(4 \wedge 12) = \mu_L(3 \vee 4) \wedge \mu_L(12)$

$\mu_L(3) \vee \mu_L(4) = \mu_L(12) \wedge \mu_L(12)$

$\mu_L(12) = \mu_L(12)$

There fore $\langle \mu_L(4), \mu_L(12) \rangle$ is an $IFMP$

Then we can take from the figure (2)

Consider $\nu_L(a, b) = \nu_L(4,12)$

Choose $\nu_L(c)$ such that

$\nu_L(c) \geq \nu_L(b)$

Choose $c = 1,2,3$

$\nu_L(c) \wedge \nu_L(a \wedge b) = \nu_L(c \vee a) \vee \nu_L(b)$ (2)

Put $c = 1$ in equation (2)

$\nu_L(1) \geq \nu_L(12)$

$\nu_L(1) \wedge \nu_L(4 \wedge 12) = \nu_L(1 \vee 4) \vee \nu_L(12)$

$\nu_L(1) \wedge \nu_L(12) = \nu_L(1) \vee \nu_L(12)$

$\nu_L(12) = \nu_L(12)$

Put $c = 2$ in equation (2)

$\nu_L(2) \geq \nu_L(12)$

$\nu_L(2) \wedge \nu_L(4 \wedge 12) = \nu_L(2 \vee 4) \vee \nu_L(12)$

$\nu_L(2) \wedge \nu_L(12) = \nu_L(2) \vee \nu_L(12)$

$\nu_L(12) \neq \nu_L(2)$

Put $c = 3$ in equation (2)

$\nu_L(3) \geq \nu_L(12)$

$\nu_L(3) \wedge \nu_L(4 \wedge 12) = \nu_L(3 \vee 4) \vee \nu_L(12)$

$\nu_L(3) \wedge \nu_L(12) = \nu_L(12) \vee \nu_L(12)$

$\nu_L(12) = \nu_L(12)$

Therefore $\langle \nu_L(4), \nu_L(12) \rangle$ is an $IFMP$

Therefore $(< \mu_L(4), \nu_L(4) >, < \mu_L(12), \nu_L(12) >)$ is not an $IFMP$.

Follow the remaining examples as above.

3.2.1 Theorem: If $\langle L, \mu_L, \nu_L \rangle$ is an $IFML$ $\mu_L(a) \leq \mu_L(b)$ and $\nu_L(a) \geq \nu_L(b)$ then $(< \mu_L(a), \nu_L(a) >, < \mu_L(b), \nu_L(b) >)$ is an $IFMP$

Proof: Given $\langle L, \mu_L, \nu_L \rangle$ is an $IFML$ and $\mu_L(a) \leq \mu_L(b)$ and $\nu_L(a) \geq \nu_L(b)$

Then $\mu_L(a \vee b) = \mu_L(b) and \ \nu_L(a \vee b) = \nu_L(a)$.

$$\mu_L(a \wedge b) = \mu_L(a) and \ \nu_L(a \wedge b) = \nu_L(b) \qquad (1)$$

To prove $(< \mu_L(a), \nu_L(a) >, < \mu_L(b), \nu_L(b) >)$ is an $IFMP$

That is to prove

$\mu_L(c) \vee \mu_L(a \wedge b) = \mu_L(c \vee a) \wedge \mu_L(b)$

$\nu_L(c) \wedge \nu_L(a \wedge b) = \nu_L(c \vee a) \vee \nu_L(b)$ (2) $for\ all$ $\mu_L(c) \leq \mu_L(b)\ and\ \nu_L(c) \geq \nu_L(b)$

Let $\mu_L(c) \leq \mu_L(b)$

Then $\mu_L(c) \vee \mu_L(a \wedge b) \geq min\{\mu_L(c), \mu_L(a \wedge b)\}$

$\geq min\{\mu_L(c), \mu_L(a)\} \quad by\ (1)$

$= \mu_L(c) \vee \mu_L(a)$

We have $\mu_L(c) \leq \mu_L(b)$, $\mu_L(a) \leq \mu_L(b)$

Then $\mu_L(c \vee a) \geq min\{\mu_L(c), \mu_L(a)\}$

$\geq min\{\mu_L(b), \mu_L(b)\}$

$= \mu_L(b)$

$\mu_L(c \vee a) \leq \mu_L(b)$

$\mu_L(c \vee a) \wedge \mu_L(b) = \mu_L(c \vee a)$

There fore $\mu_L(c) \vee \mu_L(a \wedge b) = \mu_L(c \vee a) \wedge \mu_L(b) \quad for\ all\ \mu_L(c) \leq \mu_L(b)$

Then we take equation (2)

Let $\nu_L(c) \geq \nu_L(b)$

Then $\nu_L(c) \wedge \nu_L(a \wedge b) \leq max\{\nu_L(c), \nu_L(a \wedge b)\}$

$\leq max\{\nu_L(c), \nu_L(b)\}\ by\ (1)$

$= \nu_L(c) \vee \nu_L(b)$

We have $\nu_L(c) \geq \nu_L(b)\ and\ \nu_L(a) \geq \nu_L(b)$

Then $\nu_L(c \vee a) \leq max\{\nu_L(c) \geq \nu_L(a)\}$

$\leq max\{\nu_L(b), \nu_L(b)\}$

$= \nu_L(b)$

$\nu_L(c \vee a) \geq \nu_L(b)$

$\nu_L(c \vee a) \vee \nu_L(b) = \nu_L(c \vee a)$

There fore $\nu_L(c) \wedge \nu_L(a \wedge b) = \nu_L(c \vee a) \vee \nu_L(b)\ \ for\ all\ \nu_L(c) \geq \nu_L(b)$

Hence $(\langle \mu_L(a), \nu_L(a)\rangle, \langle \mu_L(b), \nu_L(b)\rangle)$ is an $IFMP$

3.3. Example: consider an $IFL\ \ M_4$. The below figures shows that M_4 is an $IFML$.

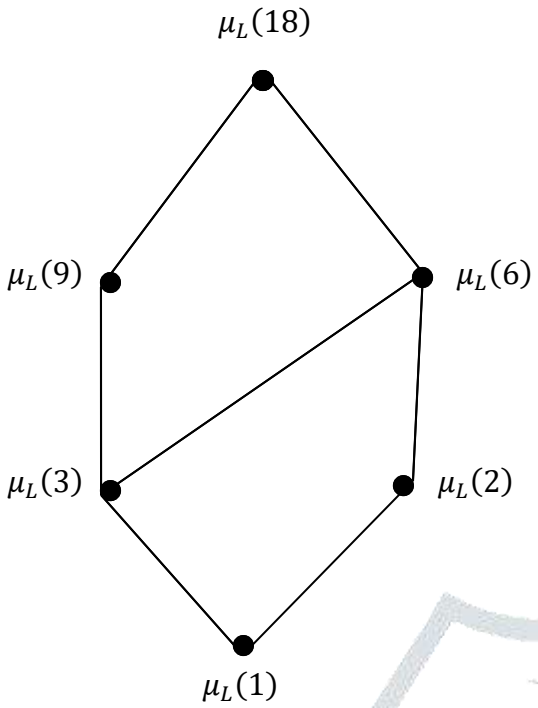

Figure 3

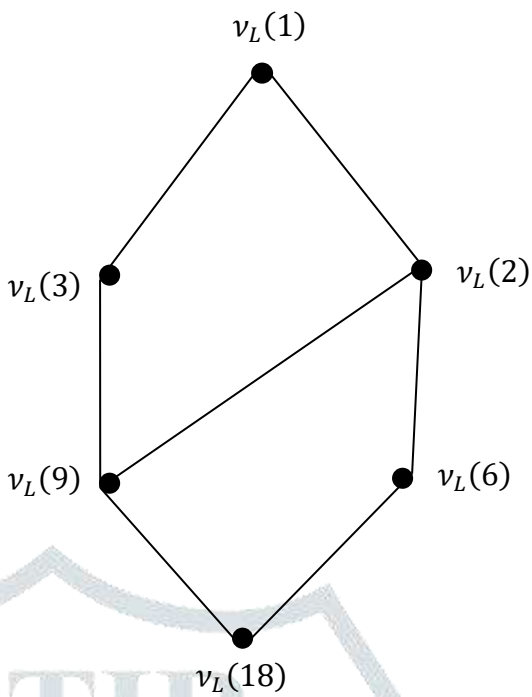

Figure 4

Consider $(a, b) = (3,9)$

Choose $\mu_L(c)$ and $\nu_L(c)$ such that $\mu_L(c) \leq \mu_L(b)$ and $\nu_L(c) \geq \nu_L(b)$

Put $c = 1,2,6$

$$\mu_L(a \vee b) \wedge \mu_L(a \vee c) = \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)] \quad (1)$$

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) = \nu_L(a) \vee [\nu_L(b) \wedge \nu_L(a \wedge c)] \quad (2)$$

Put $c = 1$ in equation (1)

$\mu_L(1) \leq \mu_L(9)$

$\mu_L(3 \vee 9) \wedge \mu_L(3 \vee 1) = \mu_L(3) \vee [\mu_L(9) \wedge \mu_L(3 \vee 1)]$

$\mu_L(9) \wedge \mu_L(3) = \mu_L(3) \vee [\mu_L(9) \wedge \mu_L(3)]$

$\mu_L(9 \wedge 3) = \mu_L(3) \vee \mu_L(3)$

$\mu_L(3) = \mu_L(3)$

Put $c = 1$ in equation (2)

$\nu_L(1) \geq \nu_L(9)$

$\nu_L(3 \vee 9) \vee \nu_L(3 \wedge 1) = \nu_L(3) \vee [\nu_L(9) \wedge \nu_L(3 \wedge 1)]$

$\nu_L(3 \vee 3) = \nu_L(3) \vee [\nu_L(9 \wedge 3)]$

$\nu_L(3) = \nu_L(3 \vee 9)$

$\nu_L(3) = \nu_L(3)$

Put $c = 2$ in equation (1)

$\mu_L(2) \leq \mu_L(9)$

$\mu_L(3 \vee 9) \wedge \mu_L(3 \vee 2) = \mu_L(3) \vee [\mu_L(9) \wedge \mu_L(3 \vee 2)]$

$\mu_L(9) \wedge \mu_L(3) = \mu_L(3) \vee [\mu_L(9 \wedge 6)]$

$\mu_L(9 \wedge 3) = \mu_L(3) \vee \mu_L(3)$

$\mu_L(3) = \mu_L(3)$

Put $c = 2$ in equation (2)

$\nu_L(2) \geq \nu_L(9)$

$\nu_L(3 \vee 9) \vee \nu_L(3 \wedge 2) = \nu_L(3) \vee [\nu_L(9) \wedge \nu_L(3 \wedge 2)]$

$\nu_L(3 \vee 6) = \nu_L(3) \vee [\nu_L(9 \wedge 6)]$

$\nu_L(3) = \nu_L(3 \vee 18)$

$\nu_L(3) = \nu_L(3)$

Put $c = 6$ in equation (1)

$\mu_L(6) \leq \mu_L(9)$

$\mu_L(3 \vee 9) \wedge \mu_L(3 \vee 6) = \mu_L(3) \vee [\mu_L(9) \wedge \mu_L(3 \vee 6)]$

$\mu_L(9) \wedge \mu_L(6) = \mu_L(3) \vee [\mu_L(9 \wedge 6)]$

$\mu_L(9 \wedge 3) = \mu_L(3) \vee \mu_L(3)$

$\mu_L(3) = \mu_L(3)$

Put $c = 6$ in equation (2)

$\nu_L(6) \geq \nu_L(9)$

$\nu_L(3 \vee 9) \vee \nu_L(3 \wedge 6) = \nu_L(3) \vee [\nu_L(9) \wedge \nu_L(3 \wedge 6)]$

$\nu_L(3 \vee 6) = \nu_L(3) \vee [\nu_L(9 \wedge 6)]$

$\nu_L(3) = \nu_L(3 \vee 18)$

$\nu_L(3) = \nu_L(3)$

Therefore $(< \mu_L(3), \nu_L(3) >, < \mu_L(9), \nu_L(9) >)$ is an $IFML$

Follow the remaining examples as above.

3.3.1 Theorem: An IFL is an $IFML$ if and only if and only if $\mu_L(a) \geq \mu_L(b)$ and $\nu_L(a) \leq \nu_L(b)$, $\mu_L(a \vee c) = \mu_L$, $\mu_L(a \wedge c) = \mu_L(b \wedge c), \nu_L(a \wedge c) = \nu_L(b \vee c), \nu_L(a \vee c) = \nu_L(b \wedge c)$ for any $\mu_L(c)$ and $\nu_L(c)$ imply that $\mu_L(a) = \mu_L(b)$ and $\nu_L(a) = \nu_L(b)$

Proof:

Assume that an IFL is an $IFML$

To prove that if $\mu_L(a) \geq \mu_L(b)$ *and* $\nu_L(a) \leq \nu_L(b)$

Then$\mu_L(a \vee c) = \mu_L(b \vee c), \mu_L(a \wedge c) = \mu_L(b \wedge c)$ (1)

$\nu_L(a \wedge c) = \nu_L(b \vee c), \nu_L(a \vee c) = \nu_L(b \wedge c)$ (2)

for any $\mu_L(c)$ *and* $\nu_L(c)$

Then $\mu_L(a) = \mu_L(b)$ and$\nu_L(a) = \nu_L(b)$.

$\mu_L(a) = \mu_L(a \wedge (a \vee c))$

$\geq min\{\mu_L(a), \mu_L(a \vee c)\}$

$\geq min\{\mu_L(a), \mu_L(b \vee c)\}$ by (1)

$\geq \{\mu_L(b \vee c), \mu_L(a)\}$

$\geq min\{\min\{\mu_L(b), \mu_L(c)\}, \mu_L(a)\}$

$\geq min\{\mu_L(b), min\{\mu_L(c), \mu_L(a)\}\}$

$\geq min\{\mu_L(b), \mu_L(c \wedge a)\}$

$\geq min\{\mu_L(b), \mu_L(b \wedge c)\}$ by (1)

$= \mu_L(b) \vee \mu_L(b \wedge c)$

$= \mu_L(b)$

Hence $\mu_L(a) = \mu_L(b)$.

$\nu_L(a) = \nu_L(a \vee (a \wedge c))$

$\leq max\{\nu_L(a), \nu_L(a \wedge c)\}$

$\leq max\{\nu_L(a), \nu_L(b \vee c)\}$ by (2)

$\leq max\{v_L(b \vee c), v_L(a)\}$

$\leq max\{max\{v_L(b), v_L(c)\}, v_L(a)\}\}$

$\leq max\{v_L(b), max\{v_L(c), v_L(a)\}\}$

$\leq max\{v_L(b), v_L(c \vee a)\}$

$\leq max\{v_L(b), v_L(b \wedge c)\}$ by (2)

$= v_L(b) \wedge v_L(b \wedge c)$

$= v_L(b)$

Hence $v_L(a) = v_L(b)$

Assume that $\langle L, \mu_L, v_L \rangle$ is an $IFML$ and if $\mu_L(a) \geq \mu_L(b), v_L(a) \leq v_L(b)$ and $\mu_L(a \vee c) = \mu_L(b \vee c), \mu_L(a \wedge c) = \mu_L(b \wedge c)$ and

$v_L(a \wedge c) = v_L(b \wedge c), v_L(a \vee c) = v_L(b \wedge c)$ for any $\mu_L(c)$ and $v_L(c)$

Then $\mu_L(a) = \mu_L(b)$ and $v_L(a) = v_L(b)$

To prove that $\langle L, \mu_L, v_L \rangle$ is an $IFML$

By definition

$\mu_L(a \vee b) \wedge \mu_L(a \vee c) = \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$ and

$v_L(a \vee b) \vee v_L(a \wedge c) = v_L(a) \vee [v_L(b) \wedge v_L(a \wedge c)]$

By assumption it is sufficient to prove

$\mu_L(a \vee b) \wedge \mu_L(a \vee c) \geq \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$

$\mu_L(a \vee b) \geq \mu_L(a)$

$\mu_L(a \vee c) \geq \mu_L(a)$

$\Rightarrow \mu_L(a \vee b) \wedge \mu_L(a \vee c) \geq min\{\mu_L(a \vee b), \mu_L(a \vee c)\}$

$\geq min\{\mu_L(a), \mu_L(a)\}$

$\geq \mu_L(a)$

$\Rightarrow \mu_L(a \vee b) \wedge \mu_L(a \vee c) \geq \mu_L(a)$ (3)

$\mu_L(a \vee b) \geq \mu_L(b)$

$$\mu_L(a \vee b) \wedge \mu_L(a \vee c) \geq min\{\mu_L(a \vee b), \mu_L(a \vee c)\} \quad (4)$$

$$\geq min\{\mu_L(b), \mu_L(a \vee c)\}$$

$$\geq \mu_L(b) \wedge \mu_L(a \vee c)$$

Equations (3) ∨ (4) gives

$$[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \vee [\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \geq \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$$

$$[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \geq min\{\mu_L(a \vee b), \mu_L(a \vee c)\}$$

$$\geq min\{\min\{\mu_L(a), \mu_L(b)\}, \mu_L(a \vee c)\}$$

$$\geq min\{\mu_L(a), min\{\mu_L(b), \mu_L(a \vee c)\}\}$$

$$\geq min\{\mu_L(a), \mu_L(b) \wedge \mu_L(a \vee c)\}$$

$$\geq \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$$

$$[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \wedge \mu_L(b) = [\mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)] \wedge \mu_L(b)$$

$$[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \vee \mu_L(b) = [\mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)] \vee \mu_L(b)$$

$$[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \wedge \mu_L(b) \geq min\{\mu_L(a \vee b) \wedge \mu_L(a \vee c), \mu_L(b)\}$$

$$\geq min\{\mu_L(a \vee c) \wedge \mu_L(a \vee b), \mu_L(b)\}$$

$$\geq min\{\mu_L(a \vee c), \mu_L(a \vee b) \wedge \mu_L(b)\}$$

$$\geq min\{\mu_L(a \vee c), \mu_L(b) \wedge \mu_L(a \vee b)\}$$

$$\geq min\{\mu_L(a \vee c), \mu_L(b) \wedge \mu_L(b \vee a)\}$$

$$\geq min\{\mu_L(a \vee c), \mu_L(b)\}$$

$$= \mu_L(a \vee c) \wedge \mu_L(b)$$

$$\mu_L(a \vee c) \wedge \mu_L(b) = [\mu_L(a \vee c) \wedge \mu_L(b)] \wedge \mu_L(b)$$

$$\leq [\mu_L(a) \vee \mu_L(b) \wedge \mu_L(a \vee c)] \wedge \mu_L(b)$$

$$\leq \mu_L(a) \vee \mu_L(a \vee c) \wedge \mu_L(b)$$

$$= \mu_L(a \vee c) \wedge \mu_L(b)$$

Therefore $\mu_L(a \vee c) \wedge \mu_L(b) = \left[\mu_L(a) \vee \left(\mu_L(b) \wedge \mu_L(a \vee c)\right)\right] \wedge \mu_L(b)$

$$\left[\mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]\right] \vee \mu_L(b) = \mu_L(a \vee b)$$

$$\mu_L(a \vee b) = \mu_L(a \vee b) \vee \mu_L(b)$$

$$\geq [\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \vee \mu_L(b)$$

$$= \left[\mu_L(b) \vee [\mu_L(c) \wedge \mu_L(a \vee b)]\right] \vee \mu_L(b)$$

$$= \mu_L(a \vee b)$$

There fore $[\mu_L(a \vee b) \wedge \mu_L(a \vee c)] \vee \mu_L(b) = \mu_L(a \vee b)$

Thus $\mu_L(a \vee b) \wedge \mu_L(a \vee c) = \mu_L(a) \vee [\mu_L(b) \wedge \mu_L(a \vee c)]$ forall $\mu_L(a), \mu_L(b), \mu_L(c) \in IFML$

By assumption it is sufficient to prove

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) \leq \nu_L(a) \vee [\nu_L(b) \wedge \nu_L(a \wedge c)]$$

$$\nu_L(a \vee b) \leq \nu_L(b)$$

$$\nu_L(b \vee c) \leq \nu_L(b)$$

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) \leq max\{\nu_L(a \vee b), \nu_L(a \wedge c)\}$$

$$\leq max\{\nu_L(b), \nu_L(b)\}$$

$$\leq \nu_L(b)$$

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) \leq \nu_L(b) \qquad (3^*)$$

$$\nu_L(a \vee b) \leq \nu_L(b)$$

$$\nu_L(a \vee b) \vee \nu_L(a \wedge c) \leq max\{\nu_L(a \vee b), \nu_L(a \wedge c)\} \qquad (4^*)$$

$$\leq max\{\nu_L(b), \nu_L(a \wedge c)\}$$

$$\leq \nu_L(b) \wedge \nu_L(a \wedge c)$$

Equations $(3^*) \vee (4^*)$ gives

$$[\nu_L(a \vee b) \vee \nu_L(a \wedge c)] \vee [\nu_L(a \vee b) \vee \nu_L(a \wedge c)] \leq \nu_L(b) \vee [\nu_L(b) \wedge \nu_L(a \wedge c)]$$

$$\nu_L(a \vee b) \vee \nu_L(a \vee c) \leq max\{\nu_L(a \vee b), \nu_L(a \wedge c)\}$$

$$\leq max\{max\{\nu_L(a), \nu_L(b)\}, \nu_L(a \wedge c)\}$$

$$\leq max\{\nu_L(a), \max\{\nu_L(b) \wedge \nu_L(a \wedge c)\}\}$$

$$\leq max\{\nu_L(a), \nu_L(b) \wedge \nu_L(a \wedge c)\}$$

$$\leq \nu_L(a) \vee [\nu_L(b) \wedge \nu_L(a \wedge c)]$$

$$[\nu_L(a \vee b) \vee \nu_L(a \wedge c)] \vee \nu_L(b) \leq max\{\nu_L(a \vee b) \vee \nu_L(a \wedge c), \nu_L(b)\}$$

$$\leq max\{\nu_L(a \wedge c) \vee \nu_L(a \vee b), \nu_L(b)\}$$

$$\leq max\{v_L(a \wedge c), v_L(a \vee b) \vee v_L(b)\}$$

$$\leq max\{v_L(a \wedge c), v_L(b) \vee v_L(a \vee b)\}$$

$$\leq max\{v_L(a \wedge c), v_L(b) \vee v_L(b \vee a)\}$$

$$\leq max\{v_L(a \wedge c), v_L(b)\}$$

$$= v_L(a \wedge c) \vee v_L(b)$$

$$v_L(a \wedge c) \vee v_L(b) = [v_L(a \wedge c) \vee v_L(b)] \vee v_L(b)$$

$$= v_L(a) \wedge \big(v_L(b) \wedge v_L(a \wedge c)\big) \vee v_L(b)$$

$$= [v_L(a) \wedge v_L(a \wedge c)] \vee v_L(b)$$

$$= v_L(a \wedge c) \vee v_L(b)$$

Therefore $v_L(a \wedge c) \vee v_L(b) = \big[v_L(a) \wedge \big(v_L(b) \wedge v_L(a \wedge c)\big)\big] \vee v_L(b)$

$$\big[v_L(a) \wedge \big(v_L(b) \wedge v_L(a \wedge c)\big)\big] \wedge v_L(b) = v_L(a \vee b)$$

$$v_L(a \vee b) = v_L(a \vee b) \wedge v_L(b)$$

$$\geq [v_L(a \wedge b) \wedge v_L(a \vee c)] \wedge v_L(b)$$

$$= \big[v_L(b) \wedge [v_L(c) \wedge v_L(a \vee b)]\big] \wedge v_L(b)$$

$$= v_L(a \vee b)$$

There fore $[v_L(a \vee b) \vee v_L(a \wedge c)] \wedge v_L(b) = v_L(a \vee b)$

Thus $v_L(a \vee b) \vee v_L(a \wedge c) = v_L(a) \vee [v_L(b) \wedge v_L(a \wedge c)]$ forall $v_L(a), v_L(b), v_L(c) \in IFML$

Hence $\langle L, \mu_{L,} v_L \rangle$ *is an IFML*

3.3.2 Theorem: If $\langle L, \mu_{L,} v_L \rangle$ *is an IFML* then $(\langle \mu_L(a), v_L(a) \rangle, \langle \mu_L(b), v_L(b) \rangle)$ *is an IFMP* for all $\mu_L(a), \mu_L(b), v_L(a), v_L(b) \in IFML$

Proof: Given $\langle L, \mu_{L,} v_L \rangle$ *is an IFML* .

To prove that $(\langle \mu_L(a), v_L(a) \rangle, \langle \mu_L(b), v_L(b) \rangle)$ *is an IFMP* for all $\mu_L(a), \mu_L(b), v_L(a), v_L(b) \in IFML$

$\Rightarrow \mu_L(x \vee y) \wedge \mu_L(x \vee z) = \mu_L(x) \vee [\mu_L(y) \wedge \mu_L(x \vee z)]$

And $v_L(x \vee y) \vee v_L(x \wedge z) = v_L(x) \vee [v_L(y) \wedge v_L(x \wedge z)]$

Take $\mu_L(c) = \mu_L(x)$, $\mu_L(a) = \mu_L(y)$, $\mu_L(b) = \mu_L(x \vee z)$

$$\mu_L(c \vee a) \wedge \mu_L(b) \geq min\{\mu_L(c \vee a), \mu_L(b)\}$$

$$\geq min\{min\{\mu_L(c), \mu_L(a)\}, \mu_L(b)\}$$

$$\geq min\{\mu_L(c), min\{\mu_L(a), \mu_L(b)\}\}$$

$$\geq min\{\mu_L(c) \vee \mu_L(a \wedge b)\} \text{ for every } \mu_L(c) \leq \mu_L(b)$$

Take $\nu_L(c) = \nu_L(x)$, $\nu_L(a) = \mu_L(y)$, $\nu_L(b) = \nu_L(x \wedge z)$

$$\nu_L(c \vee a) \vee \nu_L(b) \leq max\{\nu_L(c \vee a), \nu_L(b)\}$$

$$\leq max\{max\{\nu_L(c), \nu_L(a)\}, \nu_L(b)\}$$

$$\leq min\{\nu_L(c), max\{\nu_L(a), \nu_L(b)\}\}$$

$$\leq max\{\nu_L(c) \wedge \nu_L(a \wedge b)\} \text{ for every } \nu_L(c) \geq \nu_L(b)$$

Thus $(\langle \mu_L(a), \nu_L(a)\rangle, \langle \mu_L(b), \nu_L(b)\rangle)$ *is an IFMP*.

4. Conclusion

In this paper the definition of intuitionistic fuzzy modular pairs and intuitionistic fuzzy modular lattice was given. By using intuitionistic fuzzy modular pair we gave the conditions and characterizations of intuitionistic fuzzy modular lattices and verified them with examples.

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