Intuitionistic Fuzzy Modular Lattices: An Exploration

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Abstract: Intuitionistic fuzzy modular lattice is the generalization of fuzzy modular lattice. This paper first introduce the notions of intuitionistic fuzzy modular pairs from intuitionistic fuzzy lattices by using certain conditions and by these notions of intuitionistic fuzzy modular pairs developed intuitionistic fuzzy modular lattices with certain examples. The characterizations of intuitionistic fuzzy modular lattice were established by using intuitionistic fuzzy modular pairs.

Keywords: Intuitionistic fuzzy lattice, intuitionistic fuzzy modular pair, intuitionistic fuzzy modular lattice.

1. Introduction

Lattice has been used in Mathematics since 18th century. Modular lattice is one of the most important types of lattices. The theory of fuzzy sets proposed by Lotfi A Zadeh in 1965 has achieved a great success in various fields. With the research of fuzzy sets in 1986, K. Atanassov [1] presented intuitionistic fuzzy sets which are very effective to deal with vagueness. The concept of intuitionistic fuzzy sets is a generalization of fuzzy sets. Bustince and Burillo introduced the concept of intuitionistic fuzzy relations and investigated some of its properties and Yon and Kim introduced the notion of intuitionistic fuzzy sublattices, filters and ideals. N. Ajmal and K.V. Thomas [8] initiated such types of study in the year 1994. It was later independently established by N. Ajmal that the set of all fuzzy normal subgroups of a group constitute a sublattice of the lattice of all subgroups and is modular lattice. S.Nanda [7] proposed the notion of fuzzy lattice using the concept of fuzzy partial ordering. The concept of fuzzy lattice introduced by N.Ajmal, S.Nanda and Wilcox.L.R [7, 8, 10] explained modularity in the theory of lattices. In this paper new definitions were introduced for intuitionistic fuzzy modular lattice by using intuitionistic fuzzy modular pairs and characterization theorem for intuitionistic fuzzy modular lattice was proved.

2. Preliminaries

2.1. Definition: [K. Atanassov, 1986]: Let X be a nonempty set. An intuitionistic fuzzy set $A$ is an object having the form: $A = \{x, \mu_A(x), \nu_A(x); x \in X\}$ where the function $\mu_A: X \rightarrow [0,1]$ defines the degree of membership and $\nu_A: X \rightarrow [0,1]$ defines the degree of non-membership of each element $x \in X$ to the set $A$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$; for every $x \in X$. For convenience we can use $A = (\mu_A, \nu_A)$. 
2.1.1 Definition: [KulHur, S.Y.Jang, Y.B.Jun, 2005] Let $X$ and $Y$ be two sets. An intuitionistic fuzzy relation $R$ from $X$ to $Y$ is an intuitionistic fuzzy set of $X \times Y$ characterized by the membership function $\mu_R$ in $X \times Y$ and non-membership function $\nu_R$ in $X \times Y$ that is, $R = \{(x, y), \mu_R(x, y), \nu_R(x, y): x \in X, y \in Y\}$ an intuitionistic fuzzy relation $R$ from $X$ to $Y$ will be denoted by $R(X \times Y)$.

2.1.2 Definition: [KulHur, S.Y.Jang, Y.B.Jun, 2005] An intuitionistic fuzzy relation $R$ is

1. Reflexive if for every $x \in X$; $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$

2. Antisymmetric if for every $(x, y) \in X \times X$;

$$\mu_R(x, y) > 0 \text{ and } \mu_R(y, x) > 0 \text{ or } \nu_R(x, y) < 1 \text{ and } \nu_R(y, x) < 1 \Rightarrow x = y$$

3. Transitive if $R \circ R \subseteq R$, where $\circ$ is max-min and min-max composition, that is $\mu_R(x, z) \geq \max_y \left[ \min\{\mu_R(x, y), \mu_R(x, y)\} \right]$ and

$$\nu_R(x, z) \leq \min_y \left[ \max\{\nu_R(x, y), \nu_R(y, z)\} \right]$$

2.1.3 Definition: [K.V.Thomas, N.Ajmal] Let $L$ be a lattice and $A = \{(x, \mu_A(x), \nu_A(x))/x \in L\}$ be an intuitionistic fuzzy set of $L$. Then $A$ is called an intuitionistic fuzzy sublattice of $L$ if the following conditions are satisfied

1. $\mu_A(x \vee y) \geq \min\{\mu_A(x), \nu_A(y)\}$

2. $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \nu_A(y)\}$

3. $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$

4. $\nu_A(x \wedge y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in L$.

The set of all intuitionistic fuzzy sublattice is denoted by $IFL$.

2.1.4 Definition: [K.V.Thomas, N.Ajmal] A fuzzy subset $\mu$ of $L$ is called a fuzzy sublattice of $L$ if

1. $\mu (x \vee y) \geq \min \{\mu(x), \mu(y)\}$

2. $\mu (x \wedge y) \geq \min \{\mu(x), \mu(y)\}$, for all $x, y \in L$.

2.1.5 Definition: [Wilcox.L.R] Let $L$ be a fuzzy lattice and $\mu(a), \mu(b)$ in $L$. Thus $(\mu(a), \mu(b))$ is called a fuzzy modular pair if

$$\mu(c) \vee \mu(a \wedge b) = \mu(c \vee a) \land \mu(b), \text{for all } \mu(c) \leq \mu(b) \text{ in } L.$$ 

That is $\mu(c) \vee [\mu(a) \land \mu(c \vee b)] = \mu(c \vee a) \land \mu(c \vee b), \text{for all } \mu(c) \text{ in } L.$
3. Intuitionistic Fuzzy Modular Pairs in Intuitionistic Fuzzy Modular Lattice

3.1. Theorem: Let \( \langle L, \mu_L, \nu_L \rangle \) is an IFL on \( L \) if and only if \( \mu_L \) and \( 1 - \nu_L \) are fuzzy lattices on \( L \).

Proof: Let \( \langle L, \mu_L, \nu_L \rangle \) is an IFL on \( L \). Then by the properties

1. \( \mu_L(x \lor y) \geq \min \{ \mu_L(x), \mu_L(y) \} \)
2. \( \mu_L(x \land y) \geq \min \{ \mu_L(x), \mu_L(y) \} \)
3. \( \nu_L(x \lor y) \leq \max \{ \nu_L(x), \nu_L(y) \} \)
4. \( \nu_L(x \land y) \leq \max \{ \nu_L(x), \nu_L(y) \} \)

By condition (1) and (2) \( \mu_L \) is a fuzzy lattice.

Condition (3) gives \( 1 - \nu_L(x \lor y) \geq \min \{1 - \nu_L(x), 1 - \nu_L(y)\} \)

\[
(1 - \nu_L)(x \lor y) \geq \min((1 - \nu_L)(x),(1 - \nu_L)(y)) \tag{L_1}
\]

Condition (4) gives \( 1 - \nu_L(x \land y) \geq \min \{1 - \nu_L(x), 1 - \nu_L(y)\} \)

\[
(1 - \nu_L)(x \land y) \geq \min((1 - \nu_L)(x),(1 - \nu_L)(y)) \tag{L_2}
\]

By \( L_1 \) and \( L_2 \) \( 1 - \nu_L \) is a fuzzy lattice.

That is if \( \langle L, \mu_L, \nu_L \rangle \) is an intuitionistic fuzzy lattice.

Then \( \mu_L \) and \( 1 - \nu_L \) are fuzzy lattice.

Conversely assume that \( \mu_L \) and \( 1 - \nu_L \) are fuzzy lattice

Hence we have

1. \( \mu_L(x \lor y) \geq \min \{ \mu_L(x), \mu_L(y) \} \)
2. \( \mu_L(x \land y) \geq \min \{ \mu_L(x), \mu_L(y) \} \)
3. \( (1 - \nu_L)(x \lor y) \geq \min \{ (1 - \nu_L)(x), (1 - \nu_L)(y) \} \)
(4) \((1 - \nu_L)(x \land y) \geq \min\{(1 - \nu_L)(x), (1 - \nu_L)(y)\}\)

Then (3) becomes \(1 - \nu_L(x \lor y) \geq \min\{1 - \nu_L(x), 1 - \nu_L(y)\}\)

\(\nu_L(x \lor y) \leq \max\{\nu_L(x), \nu_L(y)\}\) \(\quad (L_3)\)

(4) becomes \(1 - \nu_L(x \land y) \geq \min\{1 - \nu_L(x), 1 - \nu_L(y)\}\)

\(\nu_L(x \land y) \leq \max\{\nu_L(x), \nu_L(y)\}\) \(\quad (L_4)\)

By (1), (2), \((L_3)\) and \((L_4)\) we get \(\langle L, \mu_L, \nu_L \rangle\) is an IFL.

3.1.1 Definition: If \(\langle \mu_L(a), \nu_L(a) \rangle\) and \(\langle \mu_L(b), \nu_L(b) \rangle\) is an intuitionistic fuzzy modular pair [IFMP] if \(\langle \mu_L(a), \mu_L(b) \rangle\) and \(\langle (1 - \nu_L)(a), (1 - \nu_L)(b) \rangle\) are fuzzy modular pair [FMP].

3.1.2 Definition: An IFL is said to be intuitionistic fuzzy modular lattice if

\[
\mu_L(a \lor b) \land \mu_L(a \lor c) = \mu_L(a) \lor [\mu_L(b) \land \mu_L(a \lor c)]
\]

\[
\nu_L(a \lor b) \lor \nu_L(a \land c) = \nu_L(a) \lor [\nu_L(b) \lor \nu_L(a \land c)]
\]

The set of all intuitionistic fuzzy modular lattices is denoted by IFML.

3.1.3 Theorem: Let \(\langle L, \mu_L, \nu_L \rangle\) is an IFML and \(\langle \mu_L(a), \nu_L(a) \rangle\) and \(\langle \mu_L(b), \nu_L(b) \rangle\) is an IFMP. Then

1) \(\mu_L(c) \lor \mu_L(a \land b) = \mu_L(c \lor a) \land \mu_L(b)\) for all \(\mu_L(c) \leq \mu_L(b)\)

2) \(\nu_L(c) \land \nu_L(a \land b) = \nu_L(c \lor a) \lor \nu_L(b)\) for all \(\nu_L(b) \leq \nu_L(c)\)

Proof: Let \(\langle L, \mu_L, \nu_L \rangle\) is an IFML and \(\langle \mu_L(a), \nu_L(a) \rangle\) and \(\langle \mu_L(b), \nu_L(b) \rangle\) is an IFMP.

Then by definition \(\langle \mu_L(a), \mu_L(b) \rangle\) and \(\langle (1 - \nu_L)(a), (1 - \nu_L)(b) \rangle\) are FMP.

Therefore \(\mu_L(c) \lor \mu_L(a \land b) = \mu_L(c \lor a) \land \mu_L(b)\) for all \(\mu_L(c) \leq \mu_L(b)\)

\((1 - \nu_L)(c) \lor (1 - \nu_L)(a \land b) = (1 - \nu_L)(c \lor a) \land (1 - \nu_L)(b)\)

\(\text{for all } (1 - \nu_L)(c) \leq (1 - \nu_L)(b)\)
That is $1 - v_L(c) \lor 1 - v_L(a \land b) = 1 - v_L(c \lor a) \land 1 - v_L(b) \quad \text{for all } 1 - v_L(c) \leq 1 - v_L(b)$

$v_L^c(c) \lor v_L^c(a \land b) = v_L^c(c \lor a) \land v_L^c(b) \quad \text{for all } v_L(c) \geq v_L(b)$

$(v_L(c) \land v_L(a \land b))^c = (v_L(c \lor a) \lor v_L(b))^c \quad \text{for all } v_L(c) \geq v_L(b)$

$v_L(c) \land v_L(a \land b) = v_L(c \lor a) \lor v_L(b) \quad \text{for all } v_L(c) \geq v_L(b)$

3.2 Example:

Consider an IFL, $N_5$ is in the following figures.

Consider $(a, b) = (4, 12)$

Choose $\mu_L(c)$ such that $\mu_L(c) \leq \mu_L(b)$

$c = 1, 2, 3$

$\mu_L(c) \lor \mu_L(a \land b) = \mu_L(c \lor a) \land \mu_L(b)$ \quad (1)

Put $c = 1$ in equation (1)

$\mu_L(1) \leq \mu_L(12)$

$\mu_L(1) \lor \mu_L(4 \land 12) = \mu_L(1 \lor 4) \land \mu_L(12)$

Figure 1

Figure 2
\[ \mu_L(1) \lor \mu_L(4) = \mu_L(4) \land \mu_L(12) \]
\[ \mu_L(4) = \mu_L(4) \]

Put \( c = 2 \) in equation (1)
\[ \mu_L(2) \leq \mu_L(12) \]
\[ \mu_L(2) \lor \mu_L(4 \land 12) = \mu_L(2 \lor 4) \land \mu_L(12) \]
\[ \mu_L(2) \lor \mu_L(4) = \mu_L(4) \land \mu_L(12) \]
\[ \mu_L(4) = \mu_L(4) \]

Put \( c = 3 \) in equation (1)
\[ \mu_L(3) \leq \mu_L(12) \]
\[ \mu_L(3) \lor \mu_L(4 \land 12) = \mu_L(3 \lor 4) \land \mu_L(12) \]
\[ \mu_L(3) \lor \mu_L(4) = \mu_L(12) \land \mu_L(12) \]
\[ \mu_L(12) = \mu_L(12) \]

Therefore \( (\mu_L(4), \mu_L(12)) \) is an IFMP

Then we can take from the figure (2)
Consider \( v_L(a, b) = v_L(4, 12) \)

Choose \( v_L(c) \) such that
\[ v_L(c) \geq v_L(b) \]

Choose \( c = 1, 2, 3 \)
\[ v_L(c) \land v_L(a \land b) = v_L(c \lor a) \lor v_L(b) \quad (2) \]

Put \( c = 1 \) in equation (2)
\[ v_L(1) \geq v_L(12) \]
\[ v_L(1) \land v_L(4 \land 12) = v_L(1 \lor 4) \lor v_L(12) \]
\[ v_L(1) \land v_L(12) = v_L(1) \lor v_L(12) \]
\[ v_L(12) = v_L(12) \]

Put \( c = 2 \) in equation (2)
\[ v_L(2) \geq v_L(12) \]
\[\nu_L(2) \land \nu_L(4 \land 12) = \nu_L(2 \lor 4) \lor \nu_L(12)\]

\[\nu_L(2) \land \nu_L(12) = \nu_L(2) \lor \nu_L(12)\]

\[\nu_L(12) \neq \nu_L(2)\]

Put \(c = 3\) in equation (2)

\[\nu_L(3) \geq \nu_L(12)\]

\[\nu_L(3) \land \nu_L(4 \land 12) = \nu_L(3 \lor 4) \lor \nu_L(12)\]

\[\nu_L(3) \land \nu_L(12) = \nu_L(12) \lor \nu_L(12)\]

\[\nu_L(12) = \nu_L(12)\]

Therefore \(\langle \nu_L(4), \nu_L(12) \rangle\) is an IFMP.

Therefore \(\langle < \mu_L(4), \nu_L(4) >, < \mu_L(12), \nu_L(12) > \rangle\) is not an IFMP.

Follow the remaining examples as above.

**3.2.1 Theorem:** If \(\langle L, \mu_L, \nu_L \rangle\) is an IFML \(\mu_L(a) \leq \mu_L(b)\) and \(\nu_L(a) \geq \nu_L(b)\), then \(\langle \mu_L(a), \nu_L(a) >, < \mu_L(b), \nu_L(b) > \rangle\) is an IFMP.

**Proof:** Given \(\langle L, \mu_L, \nu_L \rangle\) is an IFML and \(\mu_L(a) \leq \mu_L(b)\) and \(\nu_L(a) \geq \nu_L(b)\), then

\[
\mu_L(a \lor b) = \mu_L(b) \text{and } \nu_L(a \lor b) = \nu_L(a). \\
\mu_L(a \land b) = \mu_L(a) \text{and } \nu_L(a \land b) = \nu_L(b) \\
\]

(1)

To prove \(\langle < \mu_L(a), \nu_L(a) >, < \mu_L(b), \nu_L(b) > \rangle\) is an IFMP

That is to prove

\[
\mu_L(c) \lor \mu_L(a \land b) = \mu_L(c \lor a) \land \mu_L(b) \\
\nu_L(c) \land \nu_L(a \land b) = \nu_L(c \lor a) \lor \nu_L(b) \\
\mu_L(c) \leq \mu_L(b) \text{ and } \nu_L(c) \geq \nu_L(b) \\
\]

Let \(\mu_L(c) \leq \mu_L(b)\)

Then \(\mu_L(c) \lor \mu_L(a \land b) \geq \min\{ \mu_L(c), \mu_L(a \land b)\}\)

\[
\geq \min\{ \mu_L(c), \mu_L(a)\} \text{ by (1)} \\
= \mu_L(c) \lor \mu_L(a) \\
\]

We have \(\mu_L(c) \leq \mu_L(b), \mu_L(a) \leq \mu_L(b)\).
Then $\mu_L(c \lor a) \geq \min\{\mu_L(c), \mu_L(a)\}$

$$\geq \min\{\mu_L(b), \mu_L(b)\} = \mu_L(b)$$

$\mu_L(c \lor a) \leq \mu_L(b)$

$\mu_L(c \lor a) \land \mu_L(b) = \mu_L(c \lor a)$

Therefore $\mu_L(c) \lor \mu_L(a \land b) = \mu_L(c \lor a) \land \mu_L(b)$ for all $\mu_L(c) \leq \mu_L(b)$

Then we take equation (2)

Let $\nu_L(c) \geq \nu_L(b)$

Then $\nu_L(c) \land \nu_L(a \land b) \leq \max\{\nu_L(c), \nu_L(a \land b)\}$

$$\leq \max\{\nu_L(c), \nu_L(b)\} \text{ by (1)}$$

$$= \nu_L(c) \lor \nu_L(b)$$

We have $\nu_L(c) \geq \nu_L(b)$ and $\nu_L(a) \geq \nu_L(b)$

Then $\nu_L(c \lor a) \leq \max\{\nu_L(c) \geq \nu_L(a)\}$

$$\leq \max\{\nu_L(b), \nu_L(b)\} = \nu_L(b)$$

$\nu_L(c \lor a) \geq \nu_L(b)$

$\nu_L(c \lor a) \lor \nu_L(b) = \nu_L(c \lor a)$

Therefore $\nu_L(c) \land \nu_L(a \land b) = \nu_L(c \lor a) \lor \nu_L(b)$ for all $\nu_L(c) \geq \nu_L(b)$

Hence $((\mu_L(a), \nu_L(a)), (\mu_L(b), \nu_L(b)))$ is an IFMP

**3.3. Example:** consider an IFL $M_4$. The below figures shows that $M_4$ is an IFML.
Consider \((a, b) = (3, 9)\)

Choose \(\mu_L(c)\) and \(\nu_L(c)\) such that \(\mu_L(c) \leq \mu_L(b)\) and \(\nu_L(c) \geq \nu_L(b)\).

Put \(c = 1, 2, 6\)

\[
\mu_L(a \lor b) \land \mu_L(a \lor c) = \mu_L(a) \lor [\mu_L(b) \land \mu_L(a \lor c)] \tag{1}
\]

\[
\nu_L(a \lor b) \lor \nu_L(a \land c) = \nu_L(a) \lor [\nu_L(b) \lor \nu_L(a \land c)] \tag{2}
\]

Put \(c = 1\) in equation (1)

\[
\mu_L(1) \leq \mu_L(9)
\]

\[
\mu_L(3 \lor 9) \land \mu_L(3 \lor 1) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3 \lor 1)]
\]

\[
\mu_L(9) \land \mu_L(3) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3)]
\]

\[
\mu_L(9 \land 3) = \mu_L(3) \lor \mu_L(3)
\]

\[
\mu_L(3) = \mu_L(3)
\]

Put \(c = 1\) in equation (2)

\[
\nu_L(1) \geq \nu_L(9)
\]

\[
\nu_L(3 \lor 9) \lor \nu_L(3 \land 1) = \nu_L(3) \lor [\nu_L(9) \lor \nu_L(3 \land 1)]
\]

\[
\nu_L(3 \lor 3) = \nu_L(3) \lor [\nu_L(9 \land 3)]
\]
\( v_L(3) = v_L(3 \lor 9) \)
\( v_L(3) = v_L(3) \)

Put \( c = 2 \) in equation (1)

\( \mu_L(2) \leq \mu_L(9) \)
\( \mu_L(3 \lor 9) \land \mu_L(3 \lor 2) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3 \lor 2)] \)
\( \mu_L(9) \land \mu_L(3) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3)] \)
\( \mu_L(9 \land 3) = \mu_L(3) \lor \mu_L(3) \)
\( \mu_L(3) = \mu_L(3) \)

Put \( c = 2 \) in equation (2)

\( v_L(2) \geq v_L(9) \)
\( v_L(3 \lor 9) \lor v_L(3 \land 2) = v_L(3) \lor [v_L(9) \land v_L(3 \land 2)] \)
\( v_L(3 \land 6) = v_L(3) \lor [v_L(9) \land v_L(3 \land 6)] \)
\( v_L(3) = v_L(3 \lor 18) \)
\( v_L(3) = v_L(3) \)

Put \( c = 6 \) in equation (1)

\( \mu_L(6) \leq \mu_L(9) \)
\( \mu_L(3 \lor 9) \land \mu_L(3 \lor 6) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3 \lor 6)] \)
\( \mu_L(9) \land \mu_L(6) = \mu_L(3) \lor [\mu_L(9) \land \mu_L(3 \lor 6)] \)
\( \mu_L(9 \land 3) = \mu_L(3) \lor \mu_L(3) \)
\( \mu_L(3) = \mu_L(3) \)

Put \( c = 6 \) in equation (2)

\( v_L(6) \geq v_L(9) \)
\( v_L(3 \lor 9) \lor v_L(3 \land 6) = v_L(3) \lor [v_L(9) \land v_L(3 \land 6)] \)
\( v_L(3 \land 6) = v_L(3) \lor [v_L(9) \land v_L(3 \land 6)] \)
\( v_L(3) = v_L(3 \lor 18) \)
\( v_L(3) = v_L(3) \)
Therefore \( <\mu_L(3), v_L(3)> <\mu_L(9), v_L(9)> \) is an IFML.

Follow the remaining examples as above.

### 3.3.1 Theorem:

An IFL is an IFML if and only if and only if

\[
\mu_L(a) \geq \mu_L(b) \quad \text{and} \quad \nu_L(a) \leq \nu_L(b),
\]

for any \( \mu_L(c) \) and \( \nu_L(c) \).

This implies that \( \mu_L(a) = \mu_L(b) \) and \( \nu_L(a) = \nu_L(b) \).

**Proof:**

Assume that an IFL is an IFML.

To prove that if \( \mu_L(a) \geq \mu_L(b) \) and \( \nu_L(a) \leq \nu_L(b) \),

\[
\begin{align*}
\mu_L(a) &\geq \min\{\mu_L(a), \mu_L(a \lor c)\} \\
&\geq \min\{\mu_L(a), \mu_L(b \lor c)\} \quad \text{by (1)} \\
&\geq \{\mu_L(b \lor c), \mu_L(a)\} \\
&\geq \min\{\min\{\mu_L(b), \mu_L(c)\}, \mu_L(a)\} \\
&\geq \min\{\mu_L(b), \min\{\mu_L(c), \mu_L(a)\}\} \\
&\geq \min\{\mu_L(b), \mu_L(c \land a)\} \\
&\geq \min\{\mu_L(b), \mu_L(b \land c)\} \quad \text{by (1)} \\
&= \mu_L(b) \lor \mu_L(b \land c) \\
&= \mu_L(b)
\end{align*}
\]

Hence \( \mu_L(a) = \mu_L(b) \).

\[
\begin{align*}
\nu_L(a) &\leq \max\{\nu_L(a), \nu_L(a \land c)\} \\
&\leq \max\{\nu_L(a), \nu_L(b \lor c)\} \quad \text{by (2)}
\end{align*}
\]
\[
\leq \max\{\nu_L(b \lor c), \nu_L(a)\}
\]

\[
\leq \max\{\max\{\nu_L(b), \nu_L(c)\}, \nu_L(a)\}\}
\]

\[
\leq \max\{\nu_L(b), \max\{\nu_L(c), \nu_L(a)\}\}
\]

\[
\leq \max\{\nu_L(b), \nu_L(c \lor a)\}
\]

\[
\leq \max\{\nu_L(b), \nu_L(b \land c)\} \text{ by (2)}
\]

\[
= \nu_L(b) \land \nu_L(b \land c)
\]

\[
= \nu_L(b)
\]

Hence \(\nu_L(a) = \nu_L(b)\)

Assume that \(\langle L, \mu_L, \nu_L \rangle\) is an \(IFML\) and if \(\mu_L(a) \geq \mu_L(b), \nu_L(a) \leq \nu_L(b)\) and \(\mu_L(a \lor c) = \mu_L(b \lor c), \mu_L(a \land c) = \mu_L(b \land c)\) and

\(\nu_L(a \land c) = \nu_L(b \land c), \nu_L(a \lor c) = \nu_L(b \land c)\) for any \(\mu_L(c)\) and \(\nu_L(c)\)

Then \(\mu_L(a) = \mu_L(b)\) and \(\nu_L(a) = \nu_L(b)\)

To prove that \(\langle L, \mu_L, \nu_L \rangle\) is an \(IFML\)

By definition

\(\mu_L(a \lor b) \land \mu_L(a \lor c) = \mu_L(a) \lor [\mu_L(b) \land \mu_L(a \lor c)]\) and

\(\nu_L(a \lor b) \lor \nu_L(a \land c) = \nu_L(a) \lor [\nu_L(b) \lor \nu_L(a \land c)]\)

By assumption it is sufficient to prove

\(\mu_L(a \lor b) \land \mu_L(a \lor c) \geq \mu_L(a) \lor [\mu_L(b) \land \mu_L(a \lor c)]\)

\(\mu_L(a \lor b) \geq \mu_L(a)\)

\(\mu_L(a \lor c) \geq \mu_L(a)\)

\(\Rightarrow \mu_L(a \lor b) \land \mu_L(a \lor c) \geq \min\{\mu_L(a \lor b), \mu_L(a \lor c)\}\)

\[
\geq \min\{\mu_L(a), \mu_L(a)\}
\]

\[
\geq \mu_L(a)
\]

\(\Rightarrow \mu_L(a \lor b) \land \mu_L(a \lor c) \geq \mu_L(a)\) \hspace{1cm} (3)

\(\mu_L(a \lor b) \geq \mu_L(b)\)
\[ \mu_L(a ∨ b) \land \mu_L(a ∨ c) \geq \min\{\mu_L(a ∨ b), \mu_L(a ∨ c)\} \]  
\[ \geq \min\{\mu_L(b), \mu_L(a ∨ c)\} \]  
\[ \geq \mu_L(b) \land \mu_L(a ∨ c) \]

Equations (3) \lor (4) gives

\[ [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \lor [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \geq \mu_L(a) \lor [\mu_L(b) \land \mu_L(a ∨ c)] \]
\[ [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \geq \min\{\mu_L(a ∨ b), \mu_L(a ∨ c)\} \]
\[ \geq \min\{\min\{\mu_L(a), \mu_L(b)\}, \mu_L(a ∨ c)\} \]
\[ \geq \min\{\mu_L(a), \min\{\mu_L(b), \mu_L(a ∨ c)\}\} \]
\[ \geq \min\{\mu_L(a), \mu_L(b) \land \mu_L(a ∨ c)\} \]
\[ \geq \mu_L(a) \lor [\mu_L(b) \land \mu_L(a ∨ c)] \]
\[ [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \lor \mu_L(b) = [\mu_L(a) \lor [\mu_L(b) \land \mu_L(a ∨ c)] \land \mu_L(b)] \]
\[ [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \lor \mu_L(b) = [\mu_L(a) \lor [\mu_L(b) \land \mu_L(a ∨ c)] \lor \mu_L(b)] \]
\[ [\mu_L(a ∨ b) \land \mu_L(a ∨ c)] \land \mu_L(b) = \min\{\mu_L(a ∨ b) \land \mu_L(a ∨ c), \mu_L(b)\} \]
\[ \geq \min\{\mu_L(a ∨ c) \land \mu_L(a ∨ b), \mu_L(b)\} \]
\[ \geq \min\{\mu_L(a ∨ c), \mu_L(a ∨ b) \land \mu_L(b)\} \]
\[ \geq \min\{\mu_L(a ∨ c), \mu_L(b) \land \mu_L(a ∨ b)\} \]
\[ \geq \min\{\mu_L(a ∨ c), \mu_L(b) \land \mu_L(b \lor a)\} \]
\[ \geq \min\{\mu_L(a ∨ c), \mu_L(b)\} \]
\[ = \mu_L(a ∨ c) \land \mu_L(b) \]

\[ \mu_L(a ∨ c) \land \mu_L(b) = [\mu_L(a ∨ c) \land \mu_L(b)] \land \mu_L(b) \]
\[ \leq [\mu_L(a) \lor \mu_L(b) \land \mu_L(a ∨ c)] \land \mu_L(b) \]
\[ \leq \mu_L(a) \lor \mu_L(a ∨ c) \land \mu_L(b) \]
\[ = \mu_L(a ∨ c) \land \mu_L(b) \]

Therefore \[ \mu_L(a ∨ c) \land \mu_L(b) = [\mu_L(a) \lor (\mu_L(b) \land \mu_L(a ∨ c))] \land \mu_L(b) \]

\[ [\mu_L(a) \lor [\mu_L(b) \land \mu_L(a ∨ c)]] \lor \mu_L(b) = \mu_L(a ∨ b) \]

\[ \mu_L(a ∨ b) = \mu_L(a ∨ b) \lor \mu_L(b) \]
\[\begin{align*}
&\geq \mu_L(a \lor b) \land \mu_L(a \lor c) \lor \mu_L(b) \\
&= [\mu_L(b) \lor [\mu_L(c) \land \mu_L(a \lor b)]] \lor \mu_L(b) \\
&= \mu_L(a \lor b)
\end{align*}\]

Therefore \([\mu_L(a \lor b) \land \mu_L(a \lor c)] \lor \mu_L(b) = \mu_L(a \lor b)\]

Thus \(\mu_L(a \lor b) \land \mu_L(a \lor c) = \mu_L(a) \lor [\mu_L(b) \land \mu_L(a \lor c)]\) for all \(\mu_L(a), \mu_L(b), \mu_L(c) \in IFML\)

By assumption it is sufficient to prove

\[v_L(a \lor b) \lor v_L(a \land c) \leq v_L(a) \lor [v_L(b) \land v_L(a \land c)]\]

\[v_L(a \lor b) \leq v_L(b)\]

\[v_L(b \land c) \leq v_L(b)\]

\[v_L(a \lor b) \lor v_L(a \land c) \leq \max\{v_L(a \lor b), v_L(a \land c)\}\]

\[= \max\{v_L(b), v_L(b)\}\]

\[= v_L(b)\]

\[v_L(a \lor b) \lor v_L(a \land c) \leq v_L(b)\]

\[v_L(a \lor b) \leq v_L(b)\]

\[v_L(a \lor b) \lor v_L(a \land c) \leq \max\{v_L(a \lor b), v_L(a \land c)\}\]

\[= \max\{v_L(b), v_L(a \land c)\}\]

\[= v_L(b) \land v_L(a \land c)\]

Equations \((3^*) \lor (4^*)\) gives

\[\begin{align*}
&[v_L(a \lor b) \lor v_L(a \land c)] \lor [v_L(a \lor b) \lor v_L(a \land c)] \leq v_L(b) \lor [v_L(b) \land v_L(a \land c)] \\
&v_L(a \lor b) \lor v_L(a \lor c) \leq \max\{v_L(a \lor b), v_L(a \land c)\}
\end{align*}\]

\[= \max\{\max\{v_L(a), v_L(b)\}, v_L(a \land c)\}\]

\[= \max\{v_L(a), \max\{v_L(b) \land v_L(a \land c)\}\}\]

\[= \max\{v_L(a), v_L(b) \land v_L(a \land c)\}\]

\[\leq v_L(a) \lor [v_L(b) \land v_L(a \land c)]\]

\[v_L(a \lor b) \lor v_L(a \land c) \leq \max\{v_L(a \lor b) \lor v_L(a \land c), v_L(b)\}\]

\[\leq \max\{v_L(a \land c) \lor v_L(a \lor b), v_L(b)\}\]
\[
\leq \max\{v_L(a \land c), v_L(a \lor b) \lor v_L(b)\}
\]
\[
\leq \max\{v_L(a \land c), v_L(b) \lor v_L(a \lor b)\}
\]
\[
\leq \max\{v_L(a \land c), v_L(b) \lor v_L(b \lor a)\}
\]
\[
\leq \max\{v_L(a \land c), v_L(b)\}
\]
\[
v_L(a \land c) \lor v_L(b) = [v_L(a \land c) \lor v_L(b)] \lor v_L(b)
\]
\[
= v_L(a) \land (v_L(b) \land v_L(a \land c)) \lor v_L(b)
\]
\[
= [v_L(a) \land v_L(a \land c)] \lor v_L(b)
\]
\[
= v_L(a \land c) \lor v_L(b)
\]

Therefore \(v_L(a \land c) \lor v_L(b) = [v_L(a) \land (v_L(b) \land v_L(a \land c))] \lor v_L(b)\)

\[
[v_L(a) \land (v_L(b) \land v_L(a \land c))] \land v_L(b) = v_L(a \lor b)
\]

\[
v_L(a \lor b) = v_L(a \lor b) \land v_L(b)
\]
\[
\geq [v_L(a \land b) \land v_L(a \lor c)] \land v_L(b)
\]
\[
= [v_L(b) \land (v_L(c) \land v_L(a \lor b))] \land v_L(b)
\]
\[
= v_L(a \lor b)
\]

Therefore \([v_L(a \lor b) \lor v_L(a \land c)] \land v_L(b) = v_L(a \lor b)\)

Thus \(v_L(a \lor b) \lor v_L(a \land c) = v_L(a) \lor [v_L(b) \lor v_L(a \land c)]\) for all \(v_L(a), v_L(b), v_L(c) \in IFML\)

Hence \(\langle L, \mu_L, v_L \rangle\) is an IFML

**3.3.2 Theorem:** If \(\langle L, \mu_L, v_L \rangle\) is an IFML then \((\langle \mu_L(a), v_L(a) \rangle, \langle \mu_L(b), v_L(b) \rangle)\) is an IFMP for all \(\mu_L(a), \mu_L(b), v_L(a), v_L(b) \in IFML\)

**Proof:** Given \(\langle L, \mu_L, v_L \rangle\) is an IFML .

To prove that \((\langle \mu_L(a), v_L(a) \rangle, \langle \mu_L(b), v_L(b) \rangle)\) is an IFMP for all \(\mu_L(a), \mu_L(b), v_L(a), v_L(b) \in IFML\)

\[
\Rightarrow \mu_L(x \lor y) \land \mu_L(x \lor z) = \mu_L(x) \lor [\mu_L(y) \land \mu_L(x \lor z)]
\]

And \(v_L(x \lor y) \lor v_L(x \land z) = v_L(x) \lor [v_L(y) \land v_L(x \land z)]\)

Take \(\mu_L(c) = \mu_L(x) \land \mu_L(a) = \mu_L(y), \mu_L(b) = \mu_L(x \lor z)\)

\[
\mu_L(c \lor a) \land \mu_L(b) \geq \min\{\mu_L(c \lor a), \mu_L(b)\}
\]
\[
\geq \min\{\min\{\mu_L(c), \mu_L(a)\}, \mu_L(b)\}
\]
\[
\geq \min\{\mu_L(c), \min\{\mu_L(a), \mu_L(b)\}\}
\]
\[
\geq \min\{\mu_L(c) \lor \mu_L(a \land b)\} \text{ for every } \mu_L(c) \leq \mu_L(b)
\]

Take \(v_L(c) = v_L(x)\), \(v_L(a) = \mu_L(y)\), \(v_L(b) = v_L(x \land z)\)

\[
v_L(c \lor a) \lor v_L(b) \leq \max\{v_L(c \lor a), v_L(b)\}
\]
\[
\leq \max\{\max\{v_L(c), v_L(a)\}, v_L(b)\}
\]
\[
\leq \min\{v_L(c), \max\{v_L(a), v_L(b)\}\}
\]
\[
\leq \max\{v_L(c) \land v_L(a \land b)\} \text{ for every } v_L(c) \geq v_L(b)
\]

Thus \((\mu_L(a), v_L(a)), (\mu_L(b), v_L(b))\) is an IFMP.

4. Conclusion

In this paper the definition of intuitionistic fuzzy modular pairs and intuitionistic fuzzy modular lattice was given. By using intuitionistic fuzzy modular pair we gave the conditions and characterizations of intuitionistic fuzzy modular lattices and verified them with examples.

References:


