Posets and Forbidden induced subgraph of the Line graph

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Abstract: The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. As a continuation of the study of 2-colored and 3-colored diagrams we characterize some forbidden - preserving subposets of the posets whose cover-incomparability graph contains one of the forbidden induced subgraph of the line graph.

IndexTerms - Cover-incomparability graph, Blockgraph, Line graph, Poset

INTRODUCTION
Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10]. Let P = (V, ≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ∈ V we say that v covers u in P if u < v and there is no w in V with u < w < v. If u ≤ v we will sometimes say that u is below v, and that v is above u. Also, we will write u ∼ v if v covers u; and u ∼ w v if u is below v but not covered by v. By u ∼ v we denote that u and v are incomparable. Let V' be a nonempty subset of V. Then there is a natural poset Q = (V', ≤'), where u ≤' v if and only if u ≤ v for any u, v ∈ V'. The poset Q is called a subposet of P and its notation is simplified to Q = (V' : ≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is dual to a poset Q if for any x, y ∈ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G is V as its vertex set, and u ∼ v is an edge of G if u ∼ v, v ∼ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset will be called a C-I graph.

Lemma 1 [2] Let P be a poset and G be its C-I graph. Then
(i) G is connected;
(ii) vertices in an independent set of G lie on a common chain of P;
(iii) an antichain of P corresponds to a complete subgraph in G;
(iv) G contains no induced cycles of length greater than 4.

II. 3-colored diagrams
A 3-coloured diagram Q in [13] is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different - preserving subposets Q of some posets with G̃ isomorphic to the subgraph H. Let u, v, w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the preserving subposets. It is possible to have u ∼ v, u ∼ v and v ∼ w. Each case will appear as a preserving subposet of four different posets. If u ∼ v and v ∼ w in a subposet, then u ∼ v ∼ w is a chain in the subposet and u,v,w is an induced path in H. If there is either u ∼ v or v ∼ w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v, w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in preserving subposet Q of a poset P, there exists two elements u, v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain uw, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and vw are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams. 

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let H = {P | P is a poset with G̃ isomorphic to G}. Then H has a characterization by forbidden - preserving subposets.
Theorem 3: (Theorem 7.1.8, [7]) Let $G$ be a graph. Then $G$ is a line graph if and only if $G$ contains none of the nine forbidden graphs of Figure 1 as an induced subgraph.

![Figure 1: Nine Forbidden Induced Subgraphs of Line Graph](image)

Theorem 4: (Theorem 4.1,[12]) Let $P$ be a poset. Then $G_P$ is cograph if and only if $P$ contains none of $T_1, \ldots, T_7$, depicted in Figure 2, and no duals of $T_2$ and $T_5$ as $\preceq$-preserving subposet.

![Figure 2: Forbidden $\preceq$-preserving subposets for C-I cographs](image)

Theorem 5: (Theorem 4,[13]) If $P$ is a poset, then $G_P$ is cograph if and only if $P$ does not contain $T_1$ from Figure 1 and no 3-colored diagram $Q_C$ from Figure 3 and its dual are $\preceq$-preserving subposets.
We consider some subposets to be forbidden so that its C-I graphs belong to the graph family $\mathcal{F}(G_5)$ of $G_5$ in Figure 1.

### III. $\preceq$-preserving subposets of posets whose C-I graphs belong to the family $\mathcal{F}(G_5)$

We have the following theorem regarding the graph family $\mathcal{F}(G_5)$.

**Theorem 6:** If $P$ is a poset, then $G_P$ belongs to $\mathcal{F}(G_5)$ if and only if $P$ contains the 3-colored diagrams $Q_7$ and $Q_8$ from Figure 4 and $\preceq$-preserving subposets $U_1$, $U_2$ from Figure 5 and their duals.

Proof. Suppose $P$ contains 3-colored diagrams $Q_7$, $Q_8$ from Figure 4 and $\preceq$-preserving subposets $U_1$, $U_2$ from Figure 5. Then, clearly $G_P$ contains the graph from Figure 1(g) as an induced subgraph.

Conversely, suppose $G_P$ contains an induced subgraph isomorphic to $G_5$ as shown in Figure 1(g), with vertices labeled by $u$, $v$, $w$, $x$, $y$ and $z$. There are four induced $P_4$ in $G_5$ induced by vertex sets $\{u, v, w, x\}$, $\{u, v, y, x\}$, $\{z, v, w, x\}$ and $\{u, z, y, x\}$. Without loss of generality, we consider the $P_4$ induced by the vertices $u, v, w, x$ in $G_5$. Then by Theorem 5, there exists either a chain $u \preceq v \preceq w \preceq x$ in $P$ or there exists the 3-colored diagram isomorphic to $Q_7$ in $P$.

**Case (1):** The $P_4$ in $G_5$ induced by the vertices $u, v, w$ and $x$ is formed by the chain $u \preceq v \preceq w \preceq x$ in the poset $P$. Since $z$ is adjacent to $v$ in the graph $G_5$, either $y$ and $z$ are in a covering relation or these vertices are incomparable in $P$.

**Subcase (1.1):** $v \parallel z$.

Since there is a path of length two from $z$ to $w$ in $G_5$, there must be a chain from $z$ to $w$, let the chain be through the point $a$ defined by normal edges in $P$. Consider the vertex $y$ in $G_5$. There are two possibilities for $y$ with respect to $v$. Either $v \preceq y$ or $v \parallel y$ ($y \prec v$, since $w$ and $y$ are adjacent in $G_5$).

**Subcase (1.1.1):** $v \preceq y$ and $y \preceq x$.

**Subcase (1.1.2):** $v \parallel y$ and $y \parallel x$.

In the posets described by the subcases (1.1.1), (1.1.2), corresponding to the adjacency relations among the vertices $u$, $v$, $w$, $x$, $y$ and $z$ in the graph $G_5$, satisfy the 3-colored poset $Q_7$ and we are done.

**Subcase (1.1.3):** $v \parallel y$ and $y \parallel x$.

**Subcase (1.1.4):** $v \parallel y$ and $y \parallel x$.

In subcases (1.1.3) and (1.1.4), there is no chain from $u$ to $y$, but since there is a path of length two from $u$ to $y$ in $G_5$, there must be a chain from $u$ to $y$, we allow a dashed line from $u$ to $y$ through $z$, $y$ and $x$ can have both possibilities, namely $y \preceq x$ or $y \parallel x$ and hence the edge $xy$ can also be represented by red edge in the poset $P$. This situation is represented in the 3-colored diagram $Q_7$ shown in Figure 4.

**Subcase (1.2):** $z \preceq v$.

Since there is a path of length two from $u$ to $y$ in $G_5$, there must be a chain from $u$ to $y$ through the point $b$ defined by normal edges in $P$. In this case, $y$ and $x$ can have both possibilities, namely $y \preceq x$ or $y \parallel x$. This is represented by the subposets $U_1$ and $U_2$ shown in Figure 5.
Case (2): The $P_4$ in $G_5$ induced by the vertices $u, v, w$ and $x$ is formed by two chains of length 3 as in the poset $P$ as shown in Figure 2.

By Theorem 5, we have that the set \{u, v, w, x\} will form the 3-colored diagram $Q_c$ in Figure 3. Now we consider the vertices $y$ and $z$ in $G_5$ and find all the possibilities that these vertices can appear in the 3-colored diagram $Q_c$. Let the chain from $z$ to $w$ be defined by normal edges in $P$ as described in Case(1). Since there is a path of length two from $z$ to $x$ and a path of length three from $u$ to $x$ in $G_5$, there must be chains of length three from $z$ to $x$, and $u$ to $x$ in $P$. If both these chains pass through $b$ in $Q_c$, then both the vertices are in a covering relation with $b$ (x ≮ z and x ≮ u, since z is adjacent to v and w is adjacent to x in $G_5$). Otherwise, there must be dashed lines between $z$ and $x$, and $u$ and $x$ representing a chain of length 3 between $z$ and $x$, and $u$ and $x$ respectively. Similar is the case between $u$ and $y$ in the graph $G_5$. Therefore, there must be a chain of length three from $u$ to $y$ in $P$. If the chain passes through $a$, then there is a covering relation between $a$ and $y$ ($y ≮ u$ since $y$ and $w$ are adjacent in $G_5$). Otherwise there must be a dashed line between $u$ and $y$ representing a chain of length 3 between $u$ and $y$. Since $vw$, $vy$ and $zy$ are edges in $G_5$, there are three cases, either $v ≲ w$ or $v \parallel w$, $v ≲ y$ or $v \parallel y$ and $z ≲ y$ or $z \parallel y$ and hence these edges are red. From the above discussion, analyzing all the possibilities in which the vertices $y$ and $z$ can be related with the 3-colored diagram $Q_c$, it can be verified easily that we obtain the 3-colored diagram $Q_8$ in Figure 4, which is an extension of $Q_c$. Thus we have completed all the cases in which the vertices of the graph $G_5$ can appear in the poset $P$, which completes the proof of the theorem.
Figure 7: ⊲ - preserving subposets corresponding to Q8

Remarks
The number of forbidden ⊲ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. Here we characterize forbidden ⊲ - preserving subposets of G_P in Figure1 and introduce the idea of 3-colored diagrams to minimize the list of subposets.

References