STUDY OF SURVEY REPORT ON FIXED POINT THEOREMS AND THEIR APPLICATIONS

DR. SUNIL G. PURANE
Department of Mathematics
Jamkhed Mahavidyalaya , Jamkhed Dist :Ahmednagar (MS)

Abstract : In this survey paper, we collected the developmental history of fixed point theory. Some important results from beginning up to now are incorporated in this paper.

***IndexTerms* -Fixed point, approximation, topological, Boolean, zero of function**

1. INTRODUCTION

The fixed point theorem states the existence of fixed points under suitable conditions. Recall that in case $f : X \to X$ is a function, then y is a fixed point of f if fy = y is satisfied. The famous Brouwer fixed point theorem was given in 1912 [8].

The theorem states that if $f : B \to B$ is a continuous function and B is a ball in R^n , then f has a fixed point. This theorem simply guarantees the existence of a solution, but gives no information about the uniqueness and determination of the solution.

For example, if $f : [0, 1] \to [0, 1]$ is given by $f(x) = x^2$,

then f(0)= 0 and f(1) = 1, that is, f has 2 fixed points.

Several proofs of this theorem are given. Most of them are of topological in nature. A classical proof due to Birkhoff and Kellog was given in 1922, Similar classical proof was given in Linear Operators Volume 1, Dunford and Schwartz 1958.[1]

Brouwer theorem gives no information about the location of fixed points. However, effective methods have been developed to approximate the fixed points. Such tools are useful in calculating zeros of functions.

2. HISTORICAL THEORY

Theorem 1

Schauder fixed point theorem If B is a compact, convex subset of a Banach space X and $f : B \to B$ is a continuous function, then f has a fixed point [34]. The Schauder fixed point theorem has applications in approximation theory, game theory and other scientific area like engineering, economics and optimization theory.

The compactness condition on B is a very strong one and most of the problems in analysis do not have compact setting. It is natural to prove the theorem by relaxing the condition of compactness. Schauder proved the following theorem [34].

Theorem 2

If B is a closed bounded convex subset of a Banach space X and $f : B \to B$ is continuous map such that f(B) is compact, then f has a fixed point. The above theorem was generalized to locally convex topological vector spaces by Tychonoff in 1935 [37].

Theorem 3

If B is a nonempty compact convex subset of a locally convex topological vector space X and $f : B \to B$ is a continuous map, then f has a fixed point. Further extension of Tychonoff's theorem was given by Ky Fan [12]. A very interesting useful result in fixed point theory is due to Banach known as the Banach contraction principle [5].

Theorem 4

Recall that a map $f : X \to X$ is said to be a contraction map, if $d(fx, fy) \le k\, d(x, y)$, where X is a metric space, $x, y \in X$ and $0 \le k < 1$. Every contraction map is a continuous map, but a continuous map need not be a contraction map. For example,fx = x is a continuous map but it is not a contraction map. The method of successive approximation introduced by Liouville in 1837 and systematically developed by Picard in 1890 culminated in formulation by Banach known as the Banach contraction principle (BCP) is stated as below [5].

Theorem 5

If X is a complete metric space and $f : X \to X$ is a contraction map, then f has a unique fixed point or fx = x has a unique solution.

Proof

The proof of this theorem is constructive.

Let $x_{n+1} = fxn$, n = 1, 2, ... Then the sequence $\{x_n\}$ is a Cauchy sequence and converges to y in X.

It is easy to show that y = fy, that is, y is a fixed point of f. Since f is a contraction map so y is a unique fixed point. The Banach contraction principle is important as a source of existence and uniqueness theorems in different branches of sciences. This theorem provides an illustration of the unifying power of functional analytic methods and usefulness of fixed point theory in analysis.

The important feature of the Banach contraction principle is that it gives the existence, uniqueness and the sequence of the successive approximation converges to a solution of the problem. The important aspect of the result is that existence, uniqueness and determination, all are given by Banach contraction principle.

3. THEOREMS & LEMMA

Theorem 6

Let C be a closed convex subset of a Hilbert space H and $f : C \to H$ a continuous function such that $I - f$ is a non-expansive map and let $(I - f)C$ be bounded.

Then the sequence of iterates $u_{n+1} = P(I - f)u_n$,

$n = 1, 2, ..., u1 \in C$ converges to u

where u is a solution of the variational inequality $\langle fu, x - u \rangle \geq 0$ for all $x \in C$, provided that $\lim_{n\to\infty} d(un, F) = 0$, where F is the set of fixed points of $P(I - f) : C \to C$.

The VIP is also closely associated with the best approximation problem so this technique can be applied to problems in approximation theory. The following example is worth mentioning [11].

Theorem 7

Let C1 and C2 be two closed convex sets in Hilbert space H and g = P1P2 of proximity maps. Convergence of $\{x_n\}$ to a fixed point of g is guaranteed if either (i) one set is compact or (ii) one set is finite dimensional and the distance between the sets is attained. The contraction, contractive and non-expansive maps have been further extended to densifying, and 1- set contraction maps in 1969.

Several interesting results of fixed points were proved recently. A few results were proved separately for contraction maps and compact mappings (A continuous map with compact image is called a compact mapping). Both maps are densifying maps. Thus a fixed point theorem for densifying maps includes both for contraction and compact maps. If $f : B \to R_n$, then f is said to be a non-self map. Most of the fixed point theorems have been given for self-maps.

4. APPLICATION OF THEOREMS

The contraction, contractive and nonexpansive maps have been further extended to densifying, and 1- set contraction maps in 1969. Several interesting results of fixed points were proved recently [27].

A few results were proved separately for contraction maps and compact mappings (A continuous map with compact image is called a compact mapping). Both maps are densifying maps. Thus a fixed point theorem for densifying maps includes both for contraction and compact maps.

In 1966, Hartman and Stampacchia [16] gave the following interesting result in variational inequalities. The most recent result for implicit functions is due to Javid Ali and M. Imdad [2]. They introduce an implicit function to prove their results because of their versatility of deducing several contraction conditions in one go. Some new forms of implicit relations are also introduced recently in [3] and [6].

5. CONCLUSION

Recently several interesting results for sequence of iterates are used to find the solutions of the Variational Inequality Problems (VIP). In most of the cases the basic tool has been the sequence of successive approximation used in the study of fixed point theory. A good deal of work has been associated with the non0expansive maps. As the sequence of iterates for a non-expansive map need not always converge therefore several researchers have tried to give techniques for convergence of the sequence of iterates. The following result deals with the contraction maps in the study of variational inequality [24].

6. REFERENCES

[1] Agarwal, R. P., Meehan, M. and O' Regan, D., Fixed point theory and applications, Cambridge University Press 2001. Zbl 0960.54027.MR 1825411.

[2] Javid Ali and M. Imdad, An implicit function implies several contraction conditions, Sarajevo Journal of Mathematics 4 (17) (2008), 269-285. Zbl 1180.54052.MR 2483851.

[3] I. Altun and D. Turkoglu, Some fixed point theorems for weakly compatible mappings satisfying an implicit relation,TaiwanesseJ.Math. 13, 4 (2009),1291- 1301. Zbl 1194.54055.MR 2543744.

[4] Andrzej, G. and Dugundji, A., Fixed Point Theory, Springer Verlag, New York (2003). Zbl 1025.47002.MR 0117588.

[5] Banach, S., Sur les operations dans les ensembles abstritsetleur applications aux equations integrals, Fund. Math. 3 (1922), 133 - 181. JFM 48.1204.02.

[6] V. Berinde, Approximating fixed points of implicit almost contractions, HacettepeJ.Math. Statistics, 41 (1)(2012), 93-102. Zbl 06119892.MR 2976915.

[7] Border, K. C., Fixed point theorems with applications to economics and game theory, Cambridge University Press, 1985. Zbl 0558.47038. MR0790845.

[8] Brouwer, L. E. J., UberAbbi 100 S. Kumar

[9] Gohde, D., Zumprinzip der kontraktivenabbildung, Math. Nachr. 30 (1965), 251 - 258. Zbl 0127.08005.MR 0190718.

[10] Hartman, P. and Stampascchia, G., On some nonlinear elliptic differential equations, Acta Math. 115 (1966), 271 - 310. Zbl 142.38102.MR 34#6355.

[11] M. Imdad, S. Kumar and M. S. Khan, Remarks on some fixed point theorems satisfying implicit relations, RadoviMatematicki, 11(1) (2002), 135-143. Zbl 1033.54025.MR 1971330.

[12] Ishikawa, S., Fixed points by new iteration method, Proc. Amer Math Soc. 44 (1974), 147 - 150. Zbl 0286.47036.MR 336469.

[13] Kakutani, S., A generalization of Brouwer fixed point theorem, Duke Math. Journal, 8 (1941), 457 - 459. Zbl 0061.40304.MR 0004776.

[14] Kirk, W. A., A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly, 72 (1965), 1004 - 1006. Zbl 0141.32402.MR 0189009.

[15]Krasnosel'skii, M. A., Two remarks on the method of successive approximations, Uspekhi Mat. Nauk, 10: 1 (63) (1955), 123 - 127. Zbl 0064.120002.MR 0068119.

[16] Mann, W. R., Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506- 510.Zbl 0050.11603.MR 0054846.