BLOOD FLOW CHARACTERISTICS IN A STENOSED ARTERY

Dr. Subhrojyoti Debnath
Department of Mathematics
Brahmananda Keshab Chandra College
111/2 B. T. Road, Bon-Hooghly Kolkata – 700108. West Bengal

ABSTRACT

A mathematical model has been developed for studying the characteristics of blood flow in slip-flow regime through a stenosed artery. Momentum integral technique has been used to solve the problem and analytical expressions for the velocity profile, pressure gradient and skin friction are obtained. It is noticed that the slip parameter contributes a large to influence the velocity as well as pressure gradient and skin friction.

Keywords: Stenosis, Slip parameter, Intravascular plaques.

2000 MSC Classification: 76Z05

1. INTRODUCTION

In medical science, stenosis refers to as the localised narrowing in a blood vessel. Many cordiovascular diseases, particularly in mammalian arteries, are closely related to the nature of movement and the dynamic behaviour of blood vessel. Morbidity and fatality may arise in severe form of the disease. In fact, the formulation of stenosis in the lumen of artery is not yet known. However, various investigators [11, 12] emphasised that some of the major factors for the initiation and development of these vascular disease are due to formation of intravascular plaques in the impingements of ligaments and spurs on the blood vessel.

Various mathematical models have been investigated by several researchers [3, 4, 6, 7, 8, 9, 13] to explore informations related to blood flow through stenosed vessels of which the earliest work of Young [13] is of prime importance. It should be pointed out in this connection that although blood is a non-Newtonian suspension of cells in plasma, but for vessels at radius greater than 0.25 mm, blood may be considered to be a homogeneous Newtonian fluid [5]. It is worthwhile to mention that most of the aforementioned works are based on the usual assumption of the no-slip condition at the wall of the blood vessel. But, Bugliarello and Hayden [2] and Bennett [1], on the basis of their invitro experiments in studying the behaviour of red cells on the blood flow, suggested that there might be possibility of existence of slip-velocity for red cells at the vessel wall under certain conditions. It has also been demonstrated that the slip condition is real and not hypothetical. Mishra and Kar [7] developed a mathematical model to consider the blood flow characteristics through stenosed vessels, by accounting for slip-velocity at the boundary which was generalised recently by Sanyal and Karak [10] to include the effect of externally applied uniform magnetic field.

In the present paper, it is proposed to develop a mathematical model in analysing the characteristics of blood flow through a stenosed artery by accounting for the slip at the endothelium of the blood vessel. Momentum integral technique has been employed to solve the problem. Analytical expressions for the blood velocity, pressure gradient and skin-friction are obtained and the slip parameter has been found to have great effect on these quantities which are shown through graphs.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider a segment of blood vessel in which a mild stenosis has been developed and suppose that the flow is axisymmetric as well as steady and laminar. In biological situations, say, for flow of blood in an artery, sometimes it violates the conditions that are required to maintain a steady flow. If we introduce the Womersley's parameter

$$\alpha = R_0 \sqrt{\frac{\omega}{\nu}}, \qquad (1)$$

which is the ratio of the radius of the tube to the thickness of the oscillating boundary layer, then for large values of α, the flow does not become quasi-steady and when α is unity or less, the flow is regarded as quasi-steady. However McDonald [5] pointed out that for femoral artery (when $2.5 < \alpha < 3.5$), the flow becomes quasi-steady and it also holds for much smaller arteries than human femoral artery.

Let us denote the radius of the stenosed portion of the artery by $R(x)$, the depth of the stenosis by δ, slope of the stenosis by m, axial distance by x, radius of the undisturbed portion of the artery by R_0 and the total length of the artery by $2L_0$. Then using aforesaid assumptions we may represent the stenosis geometry by

$$R(x) = R_0\left[1 - \frac{\delta}{R_0}\exp\left(-\frac{m^2\varepsilon^2 x^2}{R_0^2}\right)\right], \qquad (2)$$

where $\varepsilon = \frac{R_0}{L_0}$ is the relative length of the stenosed portion. Taking $\bar{\delta} = \frac{\delta}{R_0}$ and $z = \frac{x}{L}$ we may write the above equation (2) as

$$\frac{R(x)}{R_0} = 1 - \bar{\delta}\exp\left(-\frac{m^2 z^2}{m_0^2}\right), \qquad (3)$$

where $m_0 = \frac{L_0}{L}$.

Considering the stenosis to be axially symmetric, we may construct the pictorial representation of the stenosed portion of the artery as shown in Fig-1.

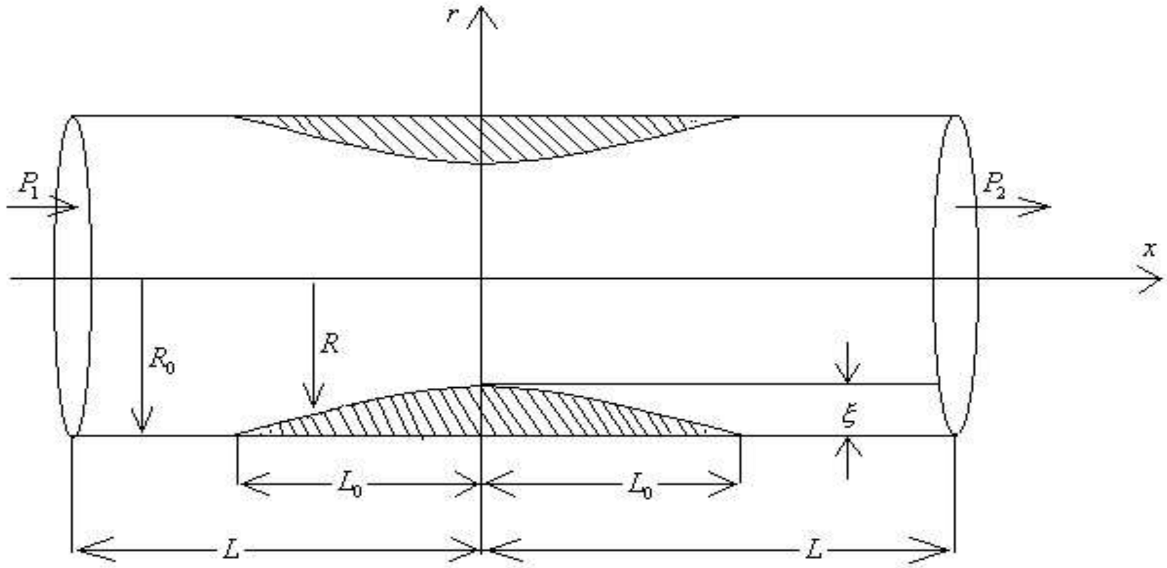


Fig. - 1 Geometrical representation of an artery with mild stenosis .

3. MATHEMATICAL ANALYSIS

Considering the flow of blood in the artery to be two-dimensional, the governing equations along the axial and radial directions are given respectively by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2}\right), \quad (4)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial x^2} - \frac{v}{r^2}\right). \quad (5)$$

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 . \quad (6)$$

In the above equations, u, v represent components of velocity along axial and radial directions, ρ is the density, p is the pressure and ν is the kinematic coefficient of viscosity of blood given by $\nu = \frac{\mu}{\rho}$, μ being the co-efficient of viscosity.

In a mild stenosis, ξ is very small with respect to L_0 and thus $\frac{\xi}{L_0}$ becomes very much lesser than unity. Thus the term $\frac{\partial^2 u}{\partial x^2}$ representing the normal stress gradient may be considered

2

∂u) to be negligible with respect to shear stress gradient $\frac{\partial^2 u}{\partial r^2}$. Also for a mild stenosis $\frac{\delta}{R_0}$ is much smaller than unity so that $\frac{\partial p}{\partial r}$ is also negligible.

Thus, the governing equations (4) and (5) may be approximated as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad (7)$$

$$\frac{\partial p}{\partial r}=0, \quad (8)$$

along with the equation of continuity (6).

Integrating equation (7) over the cross-section of the artery and using the equation of continuity (6), we obtain

$$\frac{\partial}{\partial x}\int_0^R r u^2\,dr=-\frac{R^2}{2\rho}\frac{dp}{dx}+\nu R\left(\frac{\partial u}{\partial r}\right)_{r=R}. \quad (9)$$

As there is no radial velocity at the inner wall of the artery, we have $v=0$ at $r=R$.

From the equation of continuity (6) we obtain

$$Q=\pi R^2\bar{U}=2\pi\int_0^R r u\,dr, \quad (10)$$

where Q is the volume flux, $\bar{U}$ is the mean velocity at the arbitrary cross-section of radius R. Let us choose the velocity profile as

$$\bar{u}=\frac{u}{U}=A\eta+B\eta^2+C\eta^3+D\eta^4+E, \quad (11)$$

where

$$\eta=\frac{R-r}{R}, \quad (12)$$

and U is the flow velocity at the center of the artery, A, B, C, D, E are constants which are to be determined from the boundary conditions given by

$$u=U \text{ at } r=0, \quad (13)$$

$$u=\frac{\partial u}{\partial r}=0 \text{ at } r=R, \quad (14)$$

$$\frac{\partial u}{\partial r}=0 \text{ at } r=0, \quad (15)$$

2

$$\frac{\partial^2 u}{\partial r^2} = -\frac{2U}{R^2}, \quad \text{at } r = 0 \tag{16}$$

and

$$\frac{dp}{dx} = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) \quad \text{at } r = R . \tag{17}$$

In the above, we have introduced a slip parameter β influenced by the surface flow velocity in the manner given in (14) at the inner wall of the artery. To maintain the pressure and inertial forces finite in a cylindrical fluid element the viscous force must be taken to be tending to zero, which gives $\frac{\partial u}{\partial r} = 0$ (as viscous force is proportional to $\frac{\partial u}{\partial r}$) given by (15).

If we consider the velocity profile to be nearly parabolic at the center of the artery and if this is represented by Poiseuille's profile, so that

$$\bar{u} = 1 - \left(\frac{r}{R}\right)^2$$

then the second order derivative at $r = 0$ leads to (16). At $r = R$ as $\frac{\partial u}{\partial x}$ and v become zero, equation (7) gives us (17).

If we now transfer the variable r to η by using (12), the boundary conditions (13) - (17) are reduced to

$$\left.\begin{aligned} &\bar{u} = 1 \quad at \quad \eta = 1, \\ &\bar{u} = \beta^* \frac{\partial \bar{u}}{\partial \eta} \quad at \quad \eta = 0, \\ &\frac{\partial \bar{u}}{\partial \eta} = 0 \quad at \quad \eta = 1, \\ &\frac{\partial^2 \bar{u}}{\partial \eta^2} = -2 \quad at \quad \eta = 1, \\ &\frac{dp}{dx} = \frac{\mu U}{R^2}\frac{1}{(1-\eta)}\frac{\partial}{\partial \eta}\left[(1-\eta)\frac{\partial \bar{u}}{\partial \eta}\right] \quad at \quad \eta = 0 \end{aligned}\right\} \tag{18}$$

where $\beta^* = \frac{\beta}{R}$.

Employing the aforesaid boundary conditions we may derive the expression for the velocity profile u as

$$\bar{u} = \frac{1}{\ldots}$$ [illegible]

$$\frac{1}{}\left(6^{*} 3 \ldots {}^{*}\right) 4 \frac{1}{}\left(10 \ldots {}^{*} {}^{*}\right), \qquad (19)$$

where $= -\frac{R^2}{U}\frac{dp}{dx}$ and $= 7\ 12^{*}$. (20)

Now using expression (11) and the transformation (12) we may write the volume flux from (10) as

$$Q = 2R\,U^2 \int_0^1 (1 \ldots)\, u\, d, \qquad (21)$$

which on using (19) reduces to

$$Q = \frac{R\,U^2}{30}\left(312^{*} 18 \ldots {}^{*} 2 \; 97\right).$$

Using the expression for given in (20) and taking

$$312^{*} 97, \qquad (22)$$

we get

$$U = \frac{30 Q}{R^2} \frac{\left(18^{*} 2\right)}{} \frac{R^2}{} \frac{dp}{dx}. \qquad (23)$$

To find the value of we integrate equation (9) and obtain

$$20 \ldots \frac{2}{2 \ldots 2\,U\,x} \ldots R\,U^2\, 21 \int_0^1 \ldots u\, d \ldots. \qquad (24)$$

Now we suppose that the average velocity U is

$$U = -\frac{R^2}{8}\frac{dp}{dx}. \qquad (25)$$

Obviously, $\frac{dp}{dx}$ is less than zero. So we may write

$$u = 2U\left(1 - \left(\frac{r}{R}\right)^2\right), \qquad (26)$$

on using the Poiseuille profile.

In equation (9) if we put the value of $\overline{U}$ given by (26), we obtain

$$R^2\, dp\, U \quad \frac{d}{} \quad 4\, r\overline{U}^{2} \quad \frac{R\; r}{R} \quad 2 \quad 2$$

□ 2□ *dx* □ □ □□□10□ □ *dx* 0□ □□□ R_2 □□□ *dr*

i.e.,

$$\frac{d}{dx} \square\square \frac{2}{3} R \bar{U}^2 \quad 2 \square\square \square \square R^2 \frac{dp}{dx} \square \square\bar{U} \square\square\square 10 \square \square \quad . \qquad (27)$$

Putting $\bar{U} = \frac{Q}{\square R^2}$ in this equation and then employing (23), the pressure gradient is given by

$$\frac{dp}{dx} \square\square \square \frac{8}{3} \frac{Q^2}{\square^2 R^5} \frac{dR}{dx} \square \frac{600 \square Q}{\square R^4} \square \square \,, \qquad (28)$$

where

$$\square \; \square \; \square\square \; \square \; \square \; 2\square \; \square \; 360\square^* \; \square \; 40 \quad . \qquad (29)$$

The first term on the right hand side of (28) occurs due to the inertia of the blood and the second term arises due to the viscous force. It is evident that, the slip parameter $\square^*$ influences both the inertial and viscous forces.

The non-dimensional form of the pressure-gradient is obtained from (28) as

$$\frac{\square}{\square \bar{U}_0^2} \square \frac{8 \square}{3 \square} \left(\frac{R_0}{R}\right)^5 \frac{dR}{dx} \square \frac{600 \square}{\mathrm{Re}_0} \left(\frac{R_0}{R}\right)^4 \,, \qquad (30)$$

where

$$\mathrm{Re}_0 \square \frac{\square R \bar{U}_0}{\square}_0 \,, \qquad (31)$$

is the Reynolds number and $\bar{U}_0$ is the average velocity for an undisturbed artery. The average pressure-gradient $\frac{dp}{dx}$ will be developed if

$$\frac{dp}{dx} \square 0$$

i.e., $$\frac{\square}{3} \frac{\square}{R} \frac{2 \square \mathrm{Re}_0 R_0}{} \frac{dR}{dx} \square 150 \quad . \qquad (32)$$

Putting the values of U and $\frac{dp}{dx}$ from equations (23) and (28) respectively in equation (19) we get the expression for the velocity distribution in terms of $\square$ and x as

$$\frac{u}{U_0} \square \mathrm{Re}_0 \square\square \left(\frac{R_0}{\square R}\right)^3 \square\square \frac{dR}{dx} f \square\square\square \; \square 10 \square\square \left(\frac{R_0}{\square R}\right)^2 \square\square \; g \square\square\square \,, \qquad (33)$$

where

$$f \square\square\square \; \square \; \square\square\square \; 8\square \; \square \; \square\square\square \; \square\square \; \square \; \square\square 16\square\square \; \square^* \square \; 8 \; \square \; 40\square \; \square\square\square^2$$

□ □□ 3□ □□ □□ □ □ 3□ □□

□ □□ 64□□* □ 64□□ □* □ 8 □ 32□ □□□3

□ 3□ 3□ □ 3□ □

□ □

□ □□□ 16□□ □□ □* □ 8 * □ 8 □ 32□ □□□4

□ □ □ 3□ 3□ □

□ □

80□□ □□* 8 *

□ □ (34) 3□ 3□

and

g □□□ □ □□□ 60□ □□ 30 □ 600□□□ □□□□ □ □□ □□ 360□□* □ 180□ □□ 15 □ 300□□□ □□□□□2

□ □ □ □ □ □ □ □ □□ □ 24□* 480□□*

□ □ □□ □ □ □ □□ □ 480□* 180

□ □ 36 720□ □ 3

□ 18□* 360□□* 180□* 60 12 240□ □ 4

□ □□ □ □ □ □ □ □□

□ □ □□ □ □ □ □□ □ □

□

* * * (35)

30□ 600□□ 60□

in which □ □ □ □

□ □□ □

□□18□* □ 2 .

The skin-friction □□ is given by

□ □u □ (36)

□□ □ □□ □,

□ □r □r R□

which on using equation (33) gives

4 3 □□ □□ 8□ □

□□□□□□□ R0 □□ dR □ 300 □□ 2 □ 1 □ 20□□□ R0 □ . (37)

□U 02 □ □□ 3□ □□ R □ dx Re0 □ □ □ □□ □□□□ R □□

The skin-friction must vanish at the separation and reattachment points so that

R e0 □□ R0 □□ dR □ 225 □□ 2 □ 1 □ 20□□□ . (38)

□ R □ dx 2 □□ □ 10□ □ □ □□□ □ □

After exceeding a certain value of Reynolds number, the flow becomes separated and the location of separation points in the diverging section is given by the condition that $\frac{1}{R}\frac{dR}{dx}$ is maximum so that

$$R\frac{d^2R}{dx^2} \,\square\, \square\left(\frac{dR}{dx}\right)^2 . \qquad (39)$$

Again with the help of the equation (2) we are led to the initial point of separation as

$$\square\, e^{\frac{m^2x^2}{L_0^2}} \left(1 \,\square\, 2m^2 \frac{x^2}{L_0^2}\right) \,\square\, \left(\frac{x}{L_0}\right)^2 ; \frac{\sqrt{9 \,\square\, 8\left(\frac{\square}{R_0}\right)} \,\square\, 1}{4m} \,\square\, \frac{L_0^2}{R_0^2} . \qquad (40)$$

4. NUMERICAL RESULTS AND DISCUSSIONS

We compute numerically the expressions for velocity distribution derived above to analyse the influence of the slip parameter □ on it. We take $L_0 \,\square\, 4R_0 \,\square\, 12\square$ and using this fact on equation (26) we can find the rate of variation of the non-dimensional axial velocity of blood flow in the stenosed portion of the artery. Here, we take the value of $\square^* \left(\square \frac{\square}{R}\right)$ as 0.01. As the Reynolds number of blood is very high, we take $Re_0 = 5000$ in Figures (2a - 2e) and the velocity distribution $\frac{u}{U_0}$ is plotted against □. Similarly taking $Re_0 = 8000$ Figures (3a - 3e) and taking $Re_0 = 10000$ Figures (4a - 4e) are constructed.

For the assumed model we observe from the figures that the slip parameter influences the axial velocity distribution largely. At the back of the stenosis where $\frac{x}{L_0} \,\square\, 0$, the slope of the stenosis acts as a resistance to the axial flow velocity. Due to this reason the slip parameter at the back of the stenosis acts to resist the axial flow velocity. Again at the front of the stenosis where $\frac{x}{L_0} \,\square\, 0$, the slope of the stenosis enhances the axial flow velocity and, therefore, the slip parameter at the front of the stenosis enhances the axial flow velocity. On the contrary, Mishra and Kar [7] showed that the slip velocity only enhances the axial flow velocity.

We also notice that at the back of the stenosis, the axial flow velocity decreases with the increase in Reynolds number. Again in front of the stenosis, the axial flow velocity increases with the increase in Reynolds number.

5. CONCLUSIONS

It is obvious from the above analysis that at the back of the stenosis where slip parameter acts as a resistance to the axial flow velocity reduces the axial flow velocity so largely that stagnation points and low velocity regions arise there, for which the lipid cells and platelets may get trapped at the region and enhances the growth of the stenosis. In this way a stenosis grows gradually within the lumen of the artery and at a time the artery remains completely blocked, which causes strokes (cerebral stroke, cardiac arrest etc.).

Again it is clear from our present investigation that as the Reynolds number becomes higher, the reverse flow velocity becomes more effective. Medical reports show that the Reynolds number in the large arteries (coronary artery, aorta, carotid artery etc.) is very high (greater than 5000) for which the occurrence of reverse flow in a stenosed portion of a large artery is common rather than in smaller arteries. In natural phenomena we also observe that the strokes usually occur due to the blockage of large arteries.

Thus our mathematical model gives a good reasoning for the occurrence of stroke due to gradual growth of stenosis and hence this may be useful in medical fields also.

REFERENCES

1. L. BENNETT, Red cell slip at wall in vitro, Science, **15**(1967), 1554-1556.
2. G. BUGLIARELLO, and J. W. HAYDEN, High speed microcinematographic studies of blood flow in vitro, Science, **138**(1962), 981-983.
3. D. F. YOUNG, Fluid mechanics of arterial stenosis, J. Biomechanical Engng., Trans. Am. Soc. Mech. Engrs., **101**(1979), 157-175.
4. J. H. FORRESTER and D. F. YOUNG, Flow through a converging-diverging tube and its implications in occlusive vascular disease-I, J.Biomech., **3**(1970), 297-316.
5. D. A. McDONALD, Blood flow in arteries, Edward Arnold Ltd., **133**(1974).
6. J. C. MISHRA and S. CHAKRAVARTY, Flow in arteries in the presence of stenosis, J. Biomech., **19**(1986), 907-918.
7. J. C. MISHRA and B. K. KAR, Momentum integral method for studying flow characteristics of blood through a stenosed vessel, Biorheology, **26**(1989), 23-35.
8. B. E. MORGAN and D. F. YOUNG, An integral method for the analysis of flow in arterial stenosis, Bull. Math. Bio., **36**(1974), 39-53.
9. J. PERKKIO and R. KESKINEN, On the effects of the concentration profiles of red cells on blood flow in the artery with stenosis, Bull. Math. Bio., **45**(1983), 259-267.
10. D. C. SANYAL and A. K. KARAK, The influence of magnetic field and slip velocity on blood flow through a stenosed artery, Proc. of the National Seminar on 'Recent Advances in Applied Mathematics' held at V.U.(2004), 1-7.
11. M. R. ROACH, An experimental study of the production and time course of poststenotic dilatation in the femoral and carotid arteries of adult dogs,Circulat. Res., **13**(1963), 537-551.
12. S. RODBARD, Dynamics of blood flow in stenotic vascular lesions, Am. Heart J., **72**(1966), 698704.
13. D. F. YOUNG, Effect of a time dependent stenosis on flow through a tube, J. Engng. Ind., Trans. Am. Soc. Mech. Engrs., **90**(1968), 248-254.

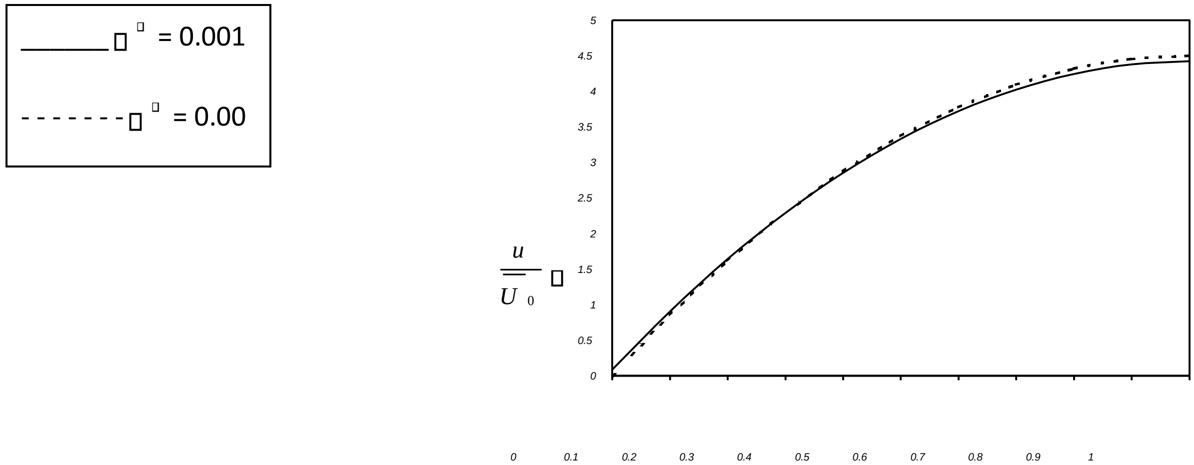


Fig. - 2a , x/L_0=0

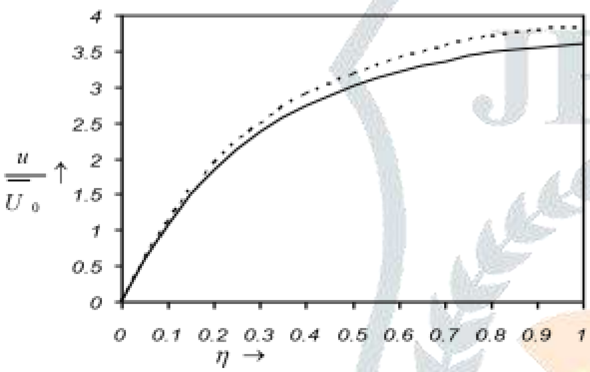


Fig. - 2b , x/L_0= -0.5

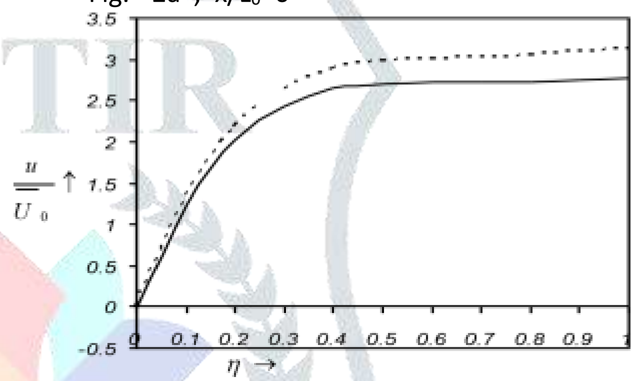


Fig. - 2c , x/L_0= -1

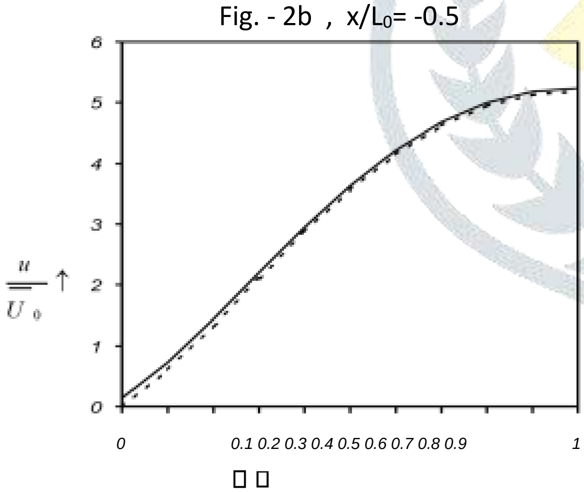


Fig. - 2d , x/L_0=0.5

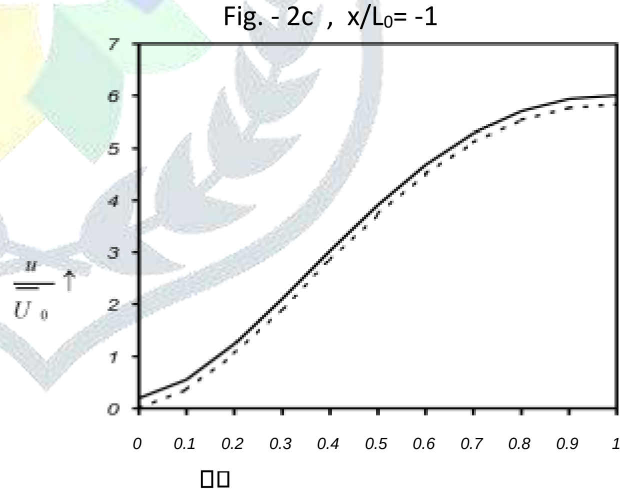
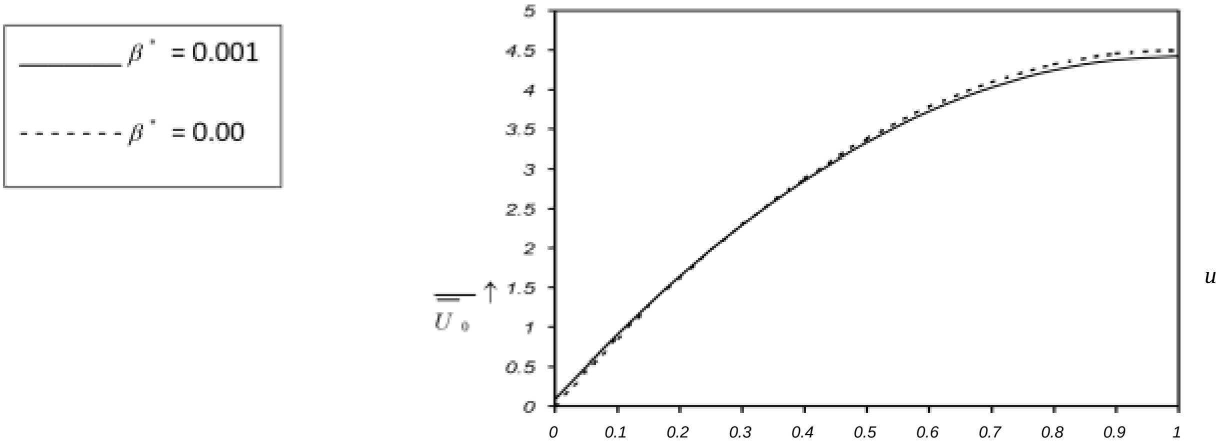
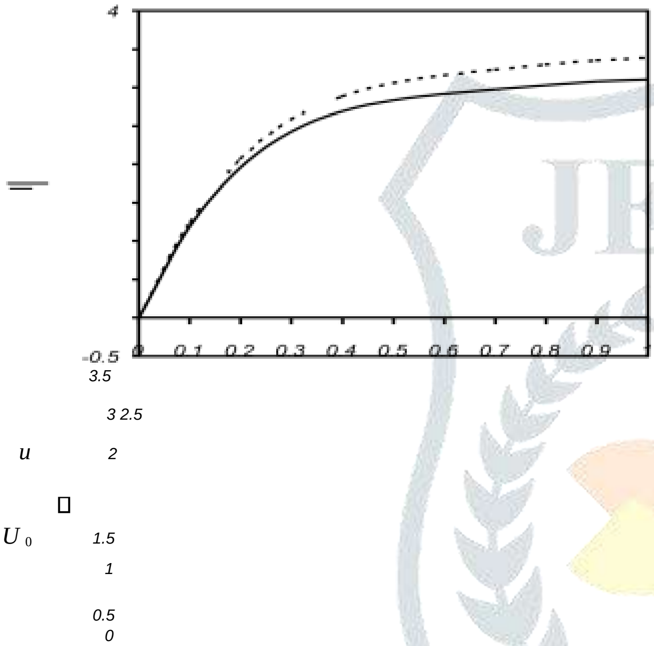


Fig. - 2e , x/L_0=1

Fig. (2a – 2e) $\frac{u}{U_0}$ versus η taking $Re_0 = 5000$



Fig, - 3a , x/L_0=0



Fig, - 3b , x/L_0= -0.5

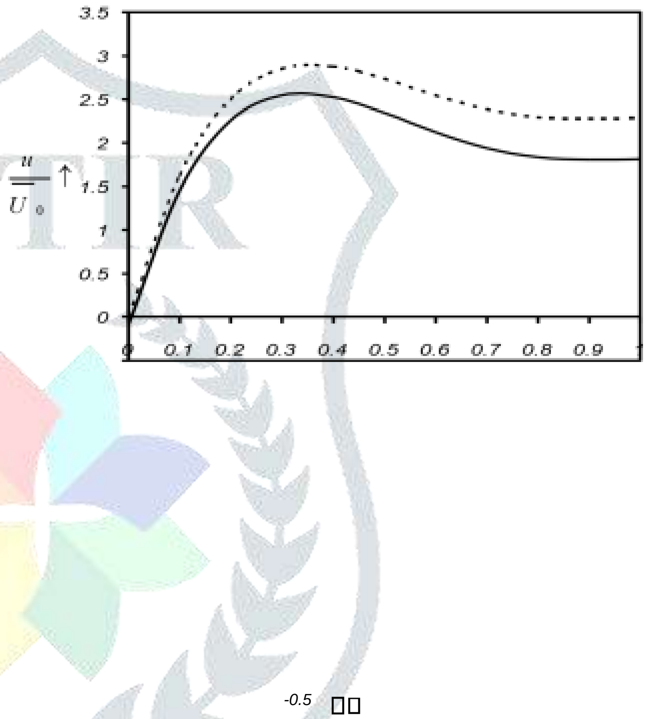


Fig. – 3c , x/L0= -1

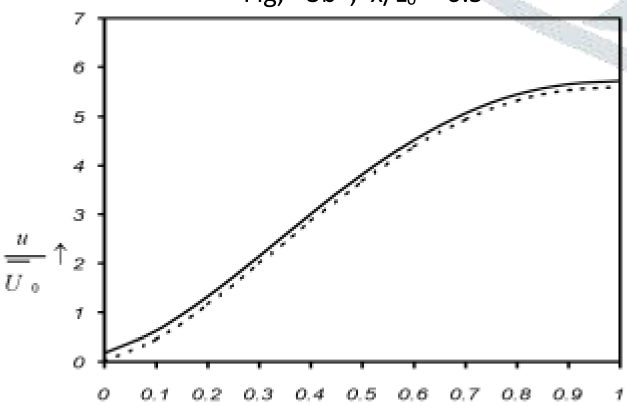


Fig. – 3d , x/L_0=0.5

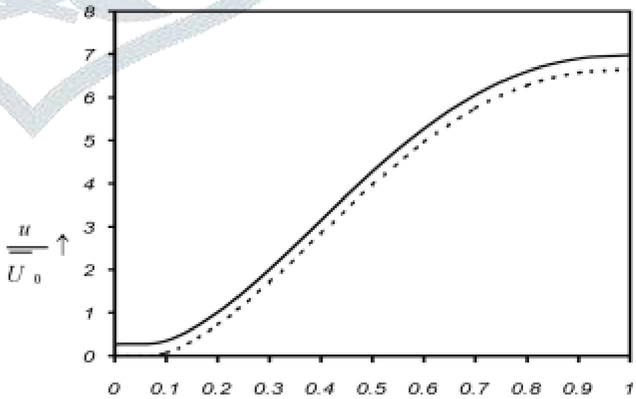
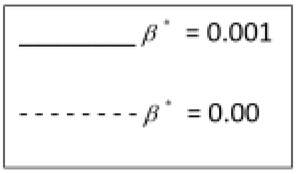


Fig. – 3e , x/L_0=1

Fig. (3a – 3e) $\frac{u}{U_0}$ versus taking $Re_0 = 8000$



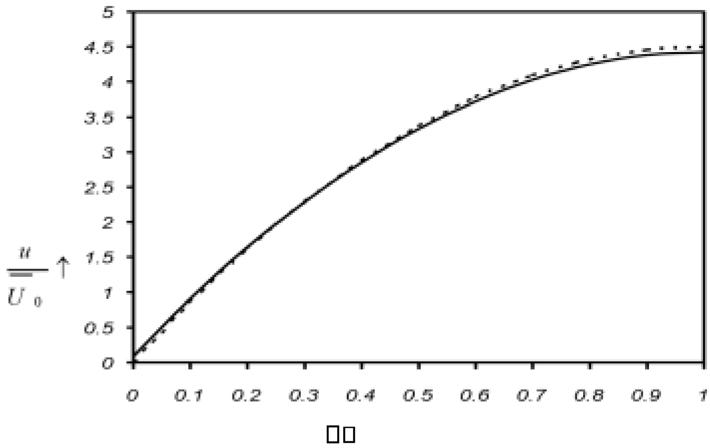


Fig. - 4a , x/L_0=0

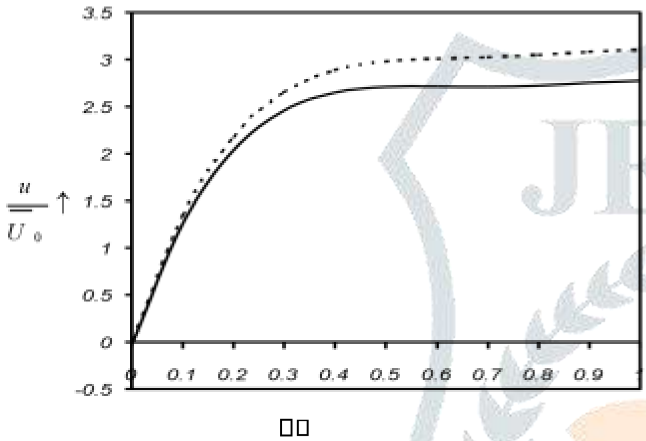


Fig. - 4b , x/L_0=-0.5

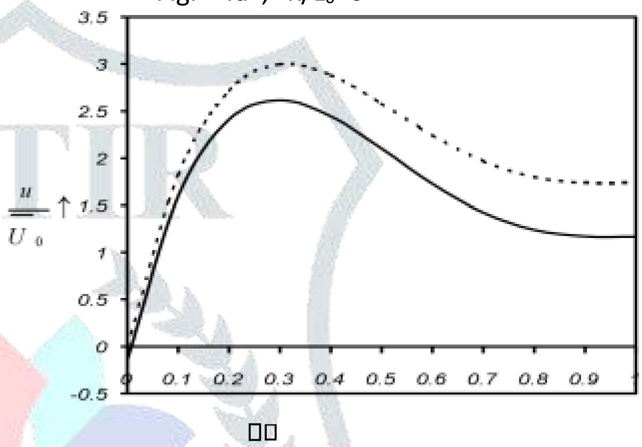


Fig. - 4c , x/L_0=-1

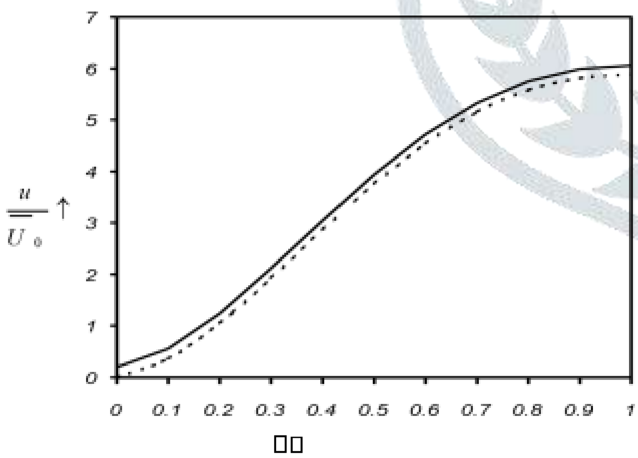


Fig. - 4d , x/L_0= 0.5

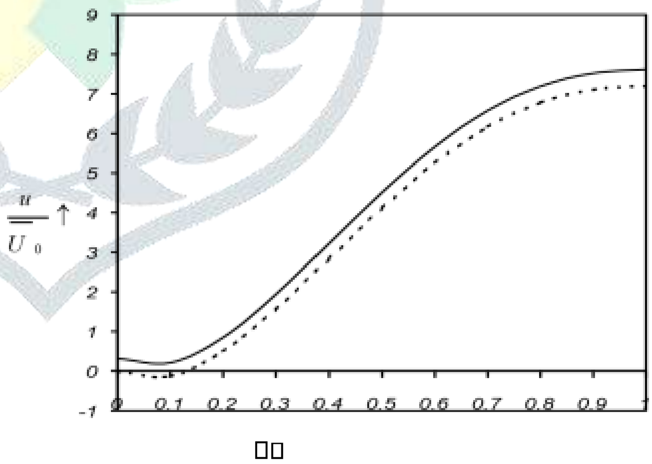


Fig. - 4e , x/L_0= 1

Fig. (4a – 4e) $\frac{u}{U_0}$ versus □ taking Re_0 = 10000