LOGISTIC GROWTH AND STABILITY ANALYSIS OF A PREDATOR-PREY MODEL WITH DISCRETE TIME DELAY

Subhransu Sundar Das[1], Dhirendra Kumar Dalai[2], Purna Chandra Nayak[3].
1. Department of Mathematics
Semiliguda College, Semiliguda, Koraput
Odisha, India
2. Department of Mathematics
Govt. (Jr) Science College, Ayeba, Kendrapara
Odisha, India
3. Department of Mathematics
Bhadrak (Auto) College, Bhadrak
Odisha, India

Abstract : The logistic growth of a predator-prey interaction model is studied by using different growth functions of prey and including a discrete time delay to model the time lags between the capture of the prey and its conversion to viable biomass.

In this paper the equilibria and stability analysis of predator-prey model is discussed, considering the different growth function of prey. If the growth function of prey is logistic then the co-existence equilibrium is locally asymptotically stable if $d_0 < \beta^n K$ and it does not exit if $d_0 > \beta^n K$. It is further shown that periodic solution is possible through Hopf-bifurcation under the conditions

$$w_0^2 > \frac{r^n d_0^2}{\beta^n k} e^{\delta\tau} \text{ and } \left[\left(1+\frac{r^n}{\beta^n k}e^{\delta\tau}\right)\left(1-\frac{2d_0}{\beta^n k}e^{\delta\tau}\right)\right]^2 > \left(\frac{d_0 e^{\delta\tau}}{\beta^n k}\right)^2$$

Key words : Biomass, Equilibrium point, Gestation period, Hopf-bifurcation, Logistic growth function, Monotonic growth function, Periodic solution, Population dynamics, Predator-prey interaction, Stability, Time delays.

Introduction :

In nature, an individual living organism of any species does not live in isolation. The organisms live in groups, are called population. Ecological studies start at the population level. Since a population changes over time, its time-rate of change is called the growth rate. The growth rate of a population is the rate of change of its size or density per unit time. It is determined by the birth-rate and the death-rate. The growth rate cannot be a constant, but it depends on the size or density of the population.

An important problem in ecology, the science which studies the interrelationship between living organisms and their environment, is to investigate the question of coexistence of two species and to decide what mankind to preserve this ecological balance of nature. In nature, there are many instances where the species of animals (the predator) feeds on another species of animals (the prey), which in turn feeds on other things that readily available in the environment. For example, population of foxes and rabbits in a woodland; the foxes (predators) eat rabbits (the prey), while the rabbits eat certain vegetation in the woodland. In the absence of the predators, the prey grows exponentially; at the same time in the absence of the prey , the predator population dies out exponentially (due to lack of food). When both predator and prey are present, there occurs, in combination with these natural rates of growth and decline is called system in equilibrium i.e, a constant population of prey and of the predator that coexist with one another in the environment. Geometrical analysis of prey-predator population follows an orbital path. Eliptic orbit are obtained around the equilibrium point or critical point.

In the layer of the planet Earth, where life exists, the growth rate of different species mainly depend on ecology and carrying capacity of environment etc. As a consequence the growth rate of prey species is an important matter for the prey-predator interaction model. The co-existence of two species has been of great interest for researchers and studied extensively using mathematical models by several researchers in particularly by [2], [3]. In many existing prey-predator models, the time delay for conversion of biomass i.e (organic material) from prey to the predator population were ignored. The delay is often caused by the conversion of consumed prey biomass into the predator biomass due to the body size growth or reproduction.

In this paper, our aim is to study and analyze the dynamics of prey-predator interacting population model due to different growth functions including discrete time τ for the capture of the prey and its conversion to biomass and the term $e^{-\delta\tau}$, which accounts for predators those interact with prey at time t but die before growth (given reproduction) τ times units later, where δ is the constant death rate for those predators that survive in gestation period i.e, the time interval between the moments when an individual prey is killed and when the corresponding biomass is added to the predator population.

For convenience, we consider the functional form

$$h(x(t)) = \beta^n x(t), \beta > 0 \text{ and } n \in R \qquad \ldots (1)$$

for both monotonic growth function and logistic growth function of prey.

In this paper our proposed model is

$$\left.\begin{aligned} \frac{d}{dt}x(t) &= g\big(x(t)\big) - h\big(x(t)\big)y(t) \\ \frac{d}{dt}y(t) &= e^{-\delta\tau}h\big(x(t-\tau)\big)y(t-\tau) - d_0 y(t) \end{aligned}\right\} \qquad \ldots (2)$$

subject to the following initial conditions :

$$\left.\begin{aligned} x(\theta) &= \phi_1(\theta), \theta \in [-\tau, 0), \phi_1(0) > 0 \\ y(\theta) &= \phi_2(\theta), \theta \in [-\tau, 0), \phi_2(0) > 0 \end{aligned}\right\} \qquad \ldots (3)$$

Here $x(t)$ denote the density of prey population, $y(t)$ is the density of the predator population, $g(x(t))$ denote the growth function of prey population, $h(x(t))$ denote the functional response of the predator on prey, d_0 is the death rate of the predator population, δ is the constant death rate for predators.

Assume that the growth rate of predator depends only on the prey population, two growth functions for the prey population are,

i) $g(x(t)) = r^n x(t), \quad r > 0$

ii) $g(x(t)) = r^n x(t)\left(1 - \dfrac{x(t)}{K}\right), \quad r > 0, \quad K > 0$

where K is the carrying capacity fo the environment

Section – I

Considering the monotonic growth function of prey and the functional form in (1), the model (2) becomes

$$\left.\begin{aligned}\frac{d}{dt}x(t) &= r^n x(t) - \beta^n x(t)y(t)\\ \frac{d}{dt}y(t) &= \beta^n e^{-\delta\tau}x(t-\tau)y(t-\tau) - d_0 y(t)\end{aligned}\right\} \quad \dots (1.1)$$

For positivity of the solution (1.1), (i.e, the predator-prey population survive) we prove the theorem following Zhu and Zou (see [12]) :

Theorem 1.1

Let $(\phi_1(\theta), \phi_2(\theta)) \in C\left([-\tau,0], R_+^2\right)$ and $\left(x(t), y(t)\right)$ be any solution to system (1.1) with the initial conditions (3), then

$$x(t) > 0, y(t) > 0 \quad \text{for } t > 0$$

Proof :

To prove $x(t) > 0$ for $t \in [0,\infty)$, from the first equation (1.1), it follows that

$$\frac{d}{dt}x(t) = r^n x(t) - \beta^n x(t)y(t)$$

$$\Rightarrow \quad \frac{d}{dt}x(t) = \left(r^n - \beta^n y(t)\right)x(t)$$

$$\Rightarrow \quad \frac{d\,x(t)}{x(t)} = \left(r^n - \beta^n y(t)\right)dt$$

On integration, we have

$$\int_0^t \frac{d\,x(t)}{x(t)} = \int_0^t \left(r^n - \beta^n y(t)\right)dt$$

$$\Rightarrow \quad \ln x(t) - \ln x(0) = \int_0^t \left(r^n - \beta^n y(t)\right)dt$$

$$\Rightarrow \quad \ln\left(\frac{x(t)}{x(0)}\right) = \int_0^t \left(r^n - \beta^n y(t)\right)dt$$

$$\Rightarrow \quad \frac{x(t)}{x(0)} = e^{\int_0^t \left(r^n - \beta^n y(t)\right)dt}$$

$$\Rightarrow \quad x(t) = x(0)\exp\left(\int_0^t r^n - \beta^n y(t)\, dt\right)$$

$$\Rightarrow \quad x(t) > 0, \text{ for all } t > 0 \qquad (\because x(0) = \phi(0) > 0)$$

Again to prove $y(t) > 0$ for all $t > 0$, if possible there exists $\bar{t} > 0$ such that $y(\bar{t}) = 0$ and $y(t) > 0$ for $t \in [0, \bar{t}]$.

Then $y'(\bar{t}) \le 0$, see [8]

But from the second equation of (1.1) , we have

$$y'(\bar{t}) = \beta^n e^{-\delta\tau} x(\bar{t}-\tau) y(\bar{t}-\tau) - dy(\bar{t})$$

$$\Rightarrow \quad y'(\bar{t}) = \beta^n e^{-\delta\tau} x(\bar{t}-\tau) y(\bar{t}-\tau) \quad (\because y(\bar{t}) = 0)$$

$$\Rightarrow \quad y'(\bar{t}) = \beta^n e^{\delta\tau} x(\bar{t}-\tau) y(\bar{t}-\tau) > 0$$, which is a contradiction,

so $y(t) > 0$, for all $t > 0$

Equilibria and Stability Analysis

We have the model (1.1) as

$$\left.\begin{aligned} \frac{d}{dt}x(t) &= r^n x(t) - \beta^n x(t) y(t) \\ \frac{d}{dt}y(t) &= \beta^n e^{-\delta\tau} x(t-\tau) y(t-\tau) - d_0\, y(t) \end{aligned}\right\}$$

we know from the predator-prey equations

$$\frac{dx}{dt} = x(a - by)$$

$$\frac{dy}{dt} = y(mx - n)$$

where a, b, m, n are positive constants; a and n are the growth rate of the prey and death rate of the predator respectively, and b and m are measures of the effect of the interaction between the two species, the critical (or equilibrium) points of the system are O(0,0) and $E\left(\frac{n}{m}, \frac{a}{b}\right)$. Here O is a trivial equilibrium and $E\left(\frac{n}{m}, \frac{a}{b}\right)$ is a non-trivial one.

The model (1.1) has two equilibrium points:

O(0,0) is a trivial equilibrium and

$E=(x^*,y^*)=\left(\frac{d_0}{e^{-\delta\tau}\beta^n},\frac{r^n}{\beta^n}\right)=\left(\frac{d_0e^{\delta\tau}}{\beta^n},\frac{r^n}{\beta^n}\right)$ is a non-trivial or co-existence equilibrium which is biologically meaningful.

Our interest is to analyze the biologically meaningful co-existence equilibrium as it specifies a constant population $\frac{d_0e^{\delta\tau}}{\beta^n}$ of prey and $\frac{r^n}{\beta^n}$ of predator that can co-exist with one another in the environment.

The linearization of (1.1) about equilibrium point (x^*,y^*) is

$$\begin{bmatrix}\frac{d}{dt}u_1(t)\\ \frac{d}{dt}u_2(t)\end{bmatrix}=\begin{bmatrix}r^n-\beta^n\bar{y} & -\beta^n\bar{x}\\ 0 & -d_0\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

$$+\begin{bmatrix}0 & 0\\ \beta^ne^{-\delta\tau}\bar{y} & \beta^ne^{-\delta\tau}\bar{x}\end{bmatrix}\begin{bmatrix}u_1(t-\tau)\\ u_2(t-\tau)\end{bmatrix} \quad \ldots (1.2)$$

$$\Rightarrow \begin{bmatrix}u_1'(t)\\ u_2'(t)\end{bmatrix}=\begin{bmatrix}(r^n-\beta^n\bar{y})u_1(t)-\beta^n\bar{x}u_2(t)\\ 0-d_0u_2(t)\end{bmatrix}+\begin{bmatrix}0\\ \beta^ne^{-\delta\tau}u_1(t-\tau)\bar{y}+\beta^ne^{-\delta\tau}u_2(t-\tau)\bar{x}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}u_1'(t)\\ u_2'(t)\end{bmatrix}=\begin{bmatrix}(r^n-\beta^n\bar{y})u_1(t)-\beta^n\bar{x}u_2(t)\\ \beta^ne^{-\delta\tau}u_1(t-\tau)\bar{y}+\beta^ne^{-\delta\tau}u_2(t-\tau)\bar{x}-d_0u_2(t)\end{bmatrix}$$

So,

$$u_1'(t)=\left(r^n-\beta^n\bar{y}\right)u_1(t)-\beta^n\bar{x}u_2(t)$$

$$u_1'(t)=\beta^ne^{-\delta\tau}\bar{y}u_1(t-\tau)+\beta^ne^{-\delta\tau}\bar{y}u_2(t-\tau)-d_0u_2(t)$$

Here the matrix

$$A=\begin{bmatrix}r^n-\beta^n\bar{y} & -\beta^n\bar{x}\\ \beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\end{bmatrix}$$

The associated characteristic equation is given by

$$\det(A-\lambda I)=0$$

$$\Rightarrow \det\left(\begin{bmatrix}r^n-\beta^n\bar{y} & -\beta^n\bar{x}\\ \beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)=0$$

$$\Rightarrow \begin{vmatrix}r^n-\beta^n\bar{y}-\lambda & -\beta^n\bar{x}\\ \beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\end{vmatrix}=0$$

$$\Rightarrow \left(r^n-\beta^n\bar{y}-\lambda\right)\left(-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\right)+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}=0$$

$$\Rightarrow \left(r^n-\beta^n\bar{y}\right)\left(-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)-\lambda\left(r^n-\beta^n\bar{y}\right)-\lambda\left(-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)+\lambda^2+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}=0$$

$$\Rightarrow \lambda^2-\left(r^n-\beta^n\bar{y}-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)\lambda+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}+\left(r^n-\beta^n\bar{y}\right)\left(-d+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0$$

Define

$$F(\lambda)=\lambda^2-(r^n-\beta^n\bar{y}-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x})\lambda+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}$$
$$+\left(r^n-\beta^n\bar{y}\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0 \qquad \dots (1.3)$$

At the equilibrium point, $E=\left(x^*,y^*\right)=\left(\frac{d_0e^{\delta\tau}}{\beta^n},\ \frac{r^n}{\beta^n}\right)$, the equation (1.3) becomes

$$F(\lambda)=\lambda^2-\left(r^n-\beta^n\frac{r^n}{\beta^n}-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\frac{d_0e^{\delta\tau}}{\beta^n}\right)\lambda$$
$$+\beta^{2n}\frac{d_0e^{\delta\tau}}{\beta^n}\frac{r^n}{\beta^n}e^{-\delta\tau}e^{-\lambda\tau}$$
$$+\left(r^n-\beta^n\frac{r^n}{\beta^n}\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\frac{d_0e^{\delta\tau}}{\beta^n}\right)=0$$

$$\Rightarrow \quad F(\lambda)=\lambda^2-\left(-d_0+d_0e^{-\lambda\tau}\right)\lambda+r^nd_0e^{-\lambda\tau}=0$$

$$\Rightarrow \quad F(\lambda)=\lambda^2+\left(d_0-d_0e^{-\lambda\tau}\right)\lambda+r^nd_0e^{-\lambda\tau}=0 \qquad \dots(1.4)$$

or, equivalently

$$F(\lambda)=\left(\lambda^2+d_0\lambda\right)+\left(-d_0\lambda+r^nd_0\right)e^{-\lambda\tau}=0$$

If $\tau=0$, then equation (1.4) becomes

$$\lambda^2+r^nd_0=0$$

$$\Rightarrow \quad \lambda^2=-r^nd_0$$

$$\Rightarrow \quad \lambda=\pm i\sqrt{r^nd_0}=\pm i\beta_0,\ \text{where}\quad \beta_0=\sqrt{r^nd_0}>0$$

We observe that, when $\tau=0$, there are no real roots and two purely imaginary roots.

Thus, it is a centre.

Now we test whether Hopf-bifurcation will occur or not

Let $G\left(r^n,\lambda\right)=\lambda^2+r^nd_0=0 \qquad \dots (1.5)$

Now, $\lambda^2+r^nd_0=0$

Differentiating both sides w.r.t r,

we get $2\lambda \frac{d\lambda}{dr} + nr^{n-1}d_0 = 0$

$$\Rightarrow \quad \frac{d\lambda}{dr} = -\frac{nr^{n-1}d_0}{2\lambda}$$

$$\therefore \quad \left.\frac{d\lambda}{dr}\right]_{\lambda=\pm i\beta_0} = \mp \frac{nr^{n-1}d_0}{2i\beta_0}$$

$$= \pm \frac{nr^{n-1}d_0}{2i\sqrt{r^n d_0}}$$

$$= \pm \frac{inr^{n-1}d_0}{2\sqrt{r^n d_0}}$$

$$= 0 \pm \frac{in\, r^{n-1}d_0}{2\sqrt{r^n d_0}}$$

$$\therefore \quad \mathrm{Re} \left.\frac{d\lambda}{dr}\right]_{\lambda=\pm i\beta_0} = 0$$

The transversality condition does not satisfy. So, if $\tau = 0$ then Hopf-bifurcation does not hold

If $\tau > 0$ then the characteristic equation for the linearized equation (1.4) around the point $E = (x^*, y^*)$ is given by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \qquad \ldots(1.6)$$

where $P(\lambda) = \lambda^2 + d_0\lambda$

$$Q(\lambda) = -d_0\lambda + r^n d_0$$

If $\lambda > 0$, let $\lambda = iw,\ w > 0$ be a purely imaginary root of equation (1.6).

From equation (1.6),

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad (\lambda^2 + d_0\lambda) + (-d_0\lambda + r^n d_0)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad \lambda^2 + (d_0 - d_0 e^{-\lambda\tau})\lambda + r^n d_0 e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad F(\lambda) = \lambda^2 + (d_0 - d_0 e^{-\lambda\tau})\lambda + r^n d_0 e^{-\lambda\tau} = 0$$

Now substituting $\lambda = iw$ in $F(\lambda)$, we get

$$F(iw) = (iw)^2 + \left(d_0 - d_0 e^{-iw\tau}\right)iw + r^n d_0 e^{-iw\tau} = 0$$

$\Rightarrow$ $$F(iw) = -w^2 + iwd_0 - iwd_0(\cos w\tau - i\sin w\tau) + r^n d_0(\cos w\tau - i\sin w\tau) = 0$$

$$\left(\because e^{i\theta} = \cos\theta + i\sin\theta\right)$$

$\Rightarrow$ $$F(iw) = \left(-w^2 - wd_0 \sin w\tau + r^n d_0 \cos w\tau\right) + i(wd_0 - wd_0 \cos w\tau - r^n d_0 \sin w\tau) = 0$$

Equating real and imaginary parts, we obtain

$$-w^2 - wd_0 \sin w\tau + r^n d_0 \cos w\tau = 0,$$

$$wd_0 - wd_0 \cos w\tau - r^n d_0 \sin w\tau = 0$$

Let $R(w) = -w^2 - wd_0 \sin w\tau + r^n d_0 \cos w\tau = 0$

and

$$S(w) = wd_0 - wd_0 \cos w\tau - r^n d_0 \sin w\tau = 0$$

$\Rightarrow$ $$wd_0 \sin w\tau - r^n d_0 \cos w\tau = -w^2 \quad \ldots (1.7)$$

$$wd_0 \cos w\tau + r^n d_0 \sin w\tau = wd_0 \quad \ldots (1.8)$$

Squaring and adding (1.8) and (1.9), we get

$$\left(wd_0 \sin w\tau - r^n d_0 \cos w\tau\right)^2 + \left(wd_0 \cos w\tau + r^n d_0 \sin w\tau\right)^2 = \left(-w^2\right)^2 + (wd_0)^2$$

$\Rightarrow$ $$w^2 d_0^2\left(\sin^2 w\tau + \cos^2 w\tau\right) + \left(r^{2n} d_0^2 (\cos^2 w\tau + \sin^2 w\tau\right) = w^4 + w^2 d_0^2$$

$\Rightarrow$ $$w^2 d_0^2 + r^{2n} d_0^2 = w^4 + w^2 d_0^2$$

$\Rightarrow$ $$w^4 - r^{2n} d_0^2 = 0$$

$\Rightarrow$ $$\left(w^2 + r^n d_0\right)\left(w^2 - r^n d_0\right) = 0$$

$\Rightarrow$ $$w^2 + r^n d_0 = 0,\ w^2 - r^n d_0 = 0$$

But $w^2 + r^n d_0 \neq 0$, as $r > 0,\ d_0 > 0,\ w > 0$

So $w^2 - r^n d_0 = 0$

$\Rightarrow$ $$w = \pm\sqrt{r^n d_0}$$

$\Rightarrow$ $$w = \sqrt{r^n d_0}, \text{ as } w > 0$$

$$\Rightarrow \quad w = w_0 = \sqrt{r^n d_0}$$

So, we have a positive $w = w_0 > 0$ such that equation (1.6) has purely imaginary roots.

Eliminating $\sin(w\tau)$ from (1.7) and (1.8) , we get

$$\sin w\tau = \frac{r^n d_0 \cos w\tau - w^2}{w d_0}, \qquad \text{from (1.7)}$$

$$\sin w\tau = \frac{w d_0 - w d_0 \cos w\tau - w^2}{r^n d_0}, \qquad \text{from (1.8)}$$

$$\text{So,} \quad \frac{r^n d_0 \cos w\tau - w^2}{w d_0} = \frac{w d_0 - w d_0 \cos w\tau}{r^n d_0}$$

$$\Rightarrow \quad r^{2n} d_0 \cos w\tau - r^n w^2 = w^2 d_0 - w^2 d_0 \cos w\tau$$

$$\Rightarrow \quad \left(r^{2n} d_0 + w^2 d_0\right) \cos w\tau = w^2 d_0 + r^n w^2$$

$$\Rightarrow \quad \cos w\tau = \frac{w^2 d_0 + r^n w^2}{r^{2n} d_0 + w^2 d_0}$$

$$\Rightarrow \quad w\tau = \cos^{-1}\left(\frac{w^2 (r^n + d_0)}{(w^2 + r^{2n}) d_0}\right)$$

$$\Rightarrow \quad \tau = \frac{1}{w} \cos^{-1}\left(\frac{w^2 (r^n + d_0)}{(w^2 + r^{2n}) d_0}\right)$$

Then τ_0 corresponding to w_0 is given by

$$\tau_0 = \frac{1}{w_0} \cos^{-1}\left(\frac{w_0^2 (r^n + d_0)}{(w_0^2 + r^{2n}) d_0}\right)$$

$$\Rightarrow \quad \tau_0 = \frac{1}{w_0}\, arc \, \cos\left(\frac{w_0^2 (r^n + d_0)}{d_0 (w_0^2 + r^{2n})}\right)$$

Hopf-bifurcation

To show $\left[\dfrac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_0} > 0$

As all roots of (1.6) depend continuously on τ and as $\tau > 0$ increases, roots of (1.6) may cross the imaginary axis only through a pair of non-zero purely imaginary roots.

For $\lambda = iw_0, w_0 > 0$ be a purely imaginary roots of (1.6),

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad P(\lambda) = -Q(\lambda)e^{-\lambda\tau}$$

$$\Rightarrow \quad |P(iw_0)| = \left|-Q(iw_0)e^{-iw_0\tau}\right|$$

$$\Rightarrow \quad |P(iw_0)| = |-Q(iw_0)||\cos w\tau - i\sin w\tau|$$

$$\Rightarrow \quad |P(iw_0)| = |Q(iw_0)| \qquad (\because |\cos w\tau - i\sin w\tau| = 1)$$

and this determines a set of possible values of λ and τ .

To determine the direction of motion of λ as λ is varied,

i.e, we determine

$$sign\left[\frac{d}{d\tau}(\operatorname{Re}\lambda)\right]_{\lambda = iw_0} = sign\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda = iw_0}$$

where sign is the signum function, defined by $\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

We have from equation (1.6),

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad (\lambda^2 + d_0\lambda) + (-d_0\lambda + r^n d_0)e^{-\lambda\tau} = 0 \qquad \ldots(1.9)$$

To find $\dfrac{d\lambda}{d\tau}$,

differentiating (1.7) with respect to τ, we get

$$\frac{d}{d\tau}\left[(\lambda^2 + d_0\lambda) + (-d_0\lambda + r^n d_0)e^{-\lambda\tau}\right] = 0$$

$$\Rightarrow \quad \frac{d}{d\tau}(\lambda^2 + d_0\lambda) + \frac{d}{d\tau}\left((-d_0\lambda + r^n d_0)e^{-\lambda\tau}\right) = 0$$

$$\Rightarrow \quad (2\lambda + d_0)\frac{d\lambda}{d\tau} + e^{-\lambda\tau}\frac{d}{d\tau}\left(-d_0\lambda + r^n d_0\right) + \left(-d_0\lambda + r^n d_0\right)\frac{d}{d\tau}\left(e^{-\lambda\tau}\right) = 0$$

$$\Rightarrow \quad (2\lambda + d_0)\frac{d\lambda}{d\tau} - d_0 e^{-\lambda\tau}\frac{d\lambda}{d\tau} + \left(-d_0\lambda + r^n d_0\right)e^{-\lambda\tau}\left(-\lambda - \tau\frac{d\lambda}{d\tau}\right) = 0$$

$$\Rightarrow \quad (2\lambda + d_0)\frac{d\lambda}{d\tau} - d_0 e^{-\lambda\tau}\frac{d\lambda}{d\tau} - \tau\left(-d_0\lambda + r^n d_0\right)e^{-\lambda\tau}\frac{d\lambda}{d\tau}$$

$$-\lambda\left(-d_0\lambda + r^n d_0\right)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad \left[(2\lambda + d_0) - d_0 e^{-\lambda\tau} - \tau\left(-d_0\lambda + r^n d_0\right)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = \lambda\left(-d_0\lambda + r^n d_0\right)e^{-\lambda\tau}$$

$$\Rightarrow \quad \frac{d\lambda}{d\tau} = \frac{\lambda\left(r^n d_0 - d_0\lambda\right)e^{-\lambda\tau}}{(2\lambda + d_0) - d_0 e^{-\lambda\tau} - \tau\left(r^n d_0 - d_0\lambda\right)e^{-\lambda\tau}}$$

$$\Rightarrow \quad \frac{d\tau}{d\lambda} = \frac{(2\lambda + d_0) - d_0 e^{-\lambda\tau} - \tau\left(r^n d_0 - d_0\lambda\right)e^{-\lambda\tau}}{\lambda\left(r^n d_0 - d_0\lambda\right)e^{-\lambda\tau}}$$

$$\Rightarrow \quad \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + d_0}{\lambda\left(r^n d_0 - d_0\lambda\right)e^{-\lambda\tau}} - \frac{d_0}{\lambda\left(r^n d_0 - d_0\lambda\right)} - \frac{\tau}{\lambda}$$

$$\Rightarrow \quad \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + d_0}{-\lambda\left(r^n d_0 - d_0\lambda\right)\left(\dfrac{\lambda^2 + d_0\lambda}{r^n d_0 - d_0\lambda}\right)} - \frac{d_0}{\lambda\left(r^n d_0 - d_0\lambda\right)} - \frac{\tau}{\lambda},$$

Using equation (1.9)

$$\Rightarrow \quad \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + d_0}{-\lambda(\lambda^2 + d_0\lambda)} - \frac{d_0}{\lambda\left(r^n d_0 - d_0\lambda\right)} - \frac{\tau}{\lambda}.$$

So,

$$\text{sign}\left[\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda = iw_0}$$

$$= sign\left[\text{Re}\left(\frac{2\lambda + d_0}{-\lambda\left(\lambda^2 + d_0\lambda\right)} - \frac{d_0}{\lambda\left(r^n d_0 - d_0\lambda\right)} - \frac{\tau}{\lambda}\right)\right]_{\lambda = iw_0}$$

$$= sign\left[\mathrm{Re}\left(\frac{2iw_0 + d_0}{-iw_0\left(i^2 w_0^2 + id_0 w_0\right)} - \frac{d_0}{iw_0\left(r^n d_0 - id_0 w_0\right)} - \frac{\tau}{iw_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{2iw_0 + d_0}{d_0 w_0^2 + iw_0^3} - \frac{d_0}{d_0 w_0^2 + ir^n d_0 w_0} - \frac{\tau}{iw_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{(2iw_0 + d_0)\left(d_0 w_0^2 - iw_0^3\right)}{\left(d_0 w_0^2\right)^2 - \left(iw_0^3\right)^2} - \frac{d_0\left(d_0 w_0^2 - ir^n d_0 w_0\right)}{(d_0 w_0)^2 - \left(ir^n d_0 w_0\right)^2} + \frac{i\tau}{w_0}\right)\right]$$

$$= sign\left[\frac{2w_0^4 + d_0^2 w_0^2}{d_0^2 w_0^4 + w_0^6} - \frac{d_0^2 w_0^2}{d_0^2 w_0^4 + r^{2n} d_0^2 w_0^2}\right]$$

$$= sign\left[\frac{2w_0^2 + d_0^2}{d_0^2 w_0^2 + w_0^4} - \frac{1}{w_0^2 + r^{2n}}\right]$$

$$= sign\left[\frac{w_0^4 + 2w_0^2 r^{2n} + r^{2n} d_0^2}{\left(d_0^2 w_0^2 + w_0^4\right)\left(w_0^2 + r^{2n}\right)}\right]$$

$$=1>0 \qquad \left(\because \frac{w_0^4 + 2w_0^2 r^{2n} + r^{2n} d_0^2}{\left(d_0^2 w_0^2 + w_0^4\right)\left(w_0^2 + r^{2n}\right)} > 0\right)$$

So,

$$sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda = iw_0} > 0$$

$$\Rightarrow \quad sign\left[\frac{d}{d\tau}(\mathrm{Re}\,\lambda)\right]_{\lambda = iw_0} > 0$$

$$\Rightarrow \quad \left[\frac{d}{d\tau}(\mathrm{Re}\lambda)\right]_{\lambda = iw_0} > 0$$

$$\Rightarrow \quad \left[\frac{d}{d\tau}(\mathrm{Re}\lambda)\right]_{w = w_0,\ \tau = \tau_0} > 0$$

From the above, we conclude that the transversality condition holds and Hopf-bifurcation occurs at $w = w_0$, $\tau = \tau_0$.

As τ increases i.e $\tau \ge \tau_0$, a periodic solution will occur which is the case of Hopf-bifurcation.

Hence if $\tau = 0$, there is a pair of purely imaginary roots and its represent centre. When τ increases to τ_0, i.e $\tau \in (0, \tau_0)$ there is another pair of purely imaginary zeros.

Section – II

Considering the logistic growth function of the prey and the functional form in (1), the model (2) becomes

$$\left.\begin{aligned}\frac{d}{dt}x(t) &= r^n x(t)\left(1-\frac{x(t)}{K}\right)-\beta^n x(t)y(t)\\ \frac{d}{dt}y(t) &= \beta^n e^{-\delta\tau}x(t-\tau)\,y(t-\tau)-d_0 y(t)\end{aligned}\right\} \qquad \ldots (2.1)$$

For the proof of positivity of the solution (2.1), we have the following theorem:

Theorem 2.1

Let $(\phi_1(\theta),\phi_2(\theta))\in C([-\tau,0],R_+^2)$ and $(x(t),y(t))$ be any solution to system (2.1) with the initial condition (3) then

$$x(t)>0,\, y(t)>0$$

Proof :

To prove $x(t)>0$ for $t\in[0,\infty)$, from the first equation in (2.1), it follows that

$$\frac{d}{dt}x(t)=x(t)\left[r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right]$$

$$\Rightarrow \quad \frac{dx(t)}{x(t)}=\left[r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right]dt$$

On integration

$$\int_0^t \frac{dx(t)}{x(t)}=\int_0^t\left[r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right]dt$$

$$\Rightarrow \quad \ln\left(\frac{x(t)}{x(0)}\right)=\int_0^t\left[r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right]dt$$

$$\Rightarrow \quad \frac{x(t)}{x(0)}=e^{\int_0^t\left[r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right]dt}$$

$$\Rightarrow \quad x(t)=x(0)\,\exp\left[\left(\int_0^t r^n-\frac{r^n}{K}x(t)-\beta^n y(t)\right)dt\right]$$

$$\Rightarrow \quad x(t)>0,\ for\ \ all\ \ t>0 \qquad (\because x(0)=\phi(0)>0)$$

Similarly, as in case of (1.1) it can be easily proved $y(t)>0$, $for\ \ all\ t>0$

Equilibria and stability Analysis

we have the model (2.1) as

$$\left.\begin{aligned}\frac{d}{dt}x(t)&=r^n x(t)\left(1-\frac{x(t)}{K}\right)-\beta^n x(t)\;y(t)\\ \frac{d}{dt}y(t)&=\beta^n e^{-\delta\tau}x(t-\tau)y(t-\tau)-d_0 y(t)\end{aligned}\right\}$$

The model (2.1) has two equilibrium points; $\overline{O}$ (0,0) is a trivial equilibrium, which is biologically meaningless and

$$\overline{E}=(\overline{x},\overline{y})=\left(\frac{d_0}{\beta^n e^{-\delta\tau}},\frac{r^n\left(1-\frac{\left(\frac{d_0}{\beta^n e^{-\delta\tau}}\right)}{K}\right)}{\beta^n}\right)$$

i.e, $\overline{E}=(\overline{x},\overline{y})=\left(\frac{d_0 e^{\delta\tau}}{\beta^n},\frac{r^n}{\beta^n}\left(1-\frac{d_0 e^{\delta\tau}}{\beta^n K}\right)\right)$ is a co-existence equilibrium which would be biologically meaningful iff

$$\frac{r^n}{\beta^n}\left(1-\frac{d_0}{\beta^n K}e^{\delta\tau}\right)>0$$

i.e, if $\left(1-\frac{d_0}{\beta^n K}e^{\delta\tau}\right)>0$

i.e, if $\beta^n k>d_0 e^{\delta\tau}$

i.e, if $e^{\delta\tau}<\frac{\beta^n K}{d_0}$

i.e, if $\delta\tau<\ln\left(\frac{\beta^n K}{d_0}\right)$

i.e, if $\tau<\frac{1}{\delta}\ln\left(\frac{\beta^n K}{d_0}\right)$

For existence of the equilibrium point, we need to be assumed $\frac{\beta^n K}{d_0}>1$.

Our interest is to analyze the biologically meaningful co-existence equilibrium.

The linearization of (2.1) about an equilibrium point $(\bar{x}, \bar{y})$ is given by

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} r^n\left(1-\frac{\bar{x}}{K}-\frac{1}{K}\bar{x}\right)-\beta^n\bar{y} & -\beta^n\bar{x} \\ 0 & -d_0 \end{bmatrix} \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 \\ \beta^n e^{-\delta\tau}\bar{y} & \beta^n e^{-\delta\tau}\bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_1(t-\tau) \\ \bar{u}_2(t-\tau) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} r^n\left(1-\frac{2\bar{x}}{K}-\beta^n\bar{y}\right) & -\beta^n\bar{x} \\ 0 & -d_0 \end{bmatrix} \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 \\ \beta^n e^{-\delta\tau}\bar{y} & \beta^n e^{-\delta\tau}\bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_1(t-\tau) \\ \bar{u}_2(t-\tau) \end{bmatrix} \qquad \ldots(2.2)$$

$$\Rightarrow \quad \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} r^n\left(1-\frac{2\bar{x}}{K}-\beta^n\bar{y}\right)\bar{u}_1(t)-\beta^n\bar{x}\bar{u}_2(t) \\ -d_0\bar{u}_2(t) \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ \beta^n e^{-\delta\tau}\bar{y}\bar{u}_1(t-\tau)+\beta^n e^{-\delta\tau}\bar{x}\bar{u}_2(t-\tau) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} \left(r^n\left(1-\frac{2\bar{x}}{K}\right)-\beta^n\bar{y}\right)\bar{u}_1(t)-\beta^n\bar{x}\bar{u}_2(t) \\ -d_0\bar{u}_2(t)+\beta^n e^{-\delta\tau}\bar{y}\bar{u}_1(t-\tau)+\beta^n e^{-\delta\tau}\bar{x}\bar{u}_2(t-\tau) \end{bmatrix}$$

Therefore,

$$u_1'(t) = \left(r^n\left(1-\frac{2\bar{x}}{K}\right)-\beta^n\bar{y}\right)\bar{u}_1(t)-\beta^n\bar{x}\bar{u}_2(t)$$

$$u_2'(t) = -d_0\bar{u}_2(t)+\beta^n e^{-\delta\tau}\bar{y}\bar{u}_1(t-\tau)+\beta^n e^{-\delta\tau}\bar{x}\bar{u}_2(t-\tau)$$

Here the matrix

$$A = \begin{bmatrix} r^n\left(1-\frac{2\bar{x}}{K}\right)-\beta^n\bar{y} & -\beta^n\bar{x} \\ \beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x} \end{bmatrix}$$

The associated characteristic equation is given by

$$\det(A-\lambda I)=0$$

$$\Rightarrow \quad \det\left(\begin{bmatrix} r^n\left(1-\frac{2\bar{x}}{K}\right)-\beta^n\bar{y} & -\beta^n\bar{x} \\ \beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)=0$$

$$\Rightarrow \quad \begin{vmatrix} r^n\left(1-\frac{2\bar{x}}{K}\right)-\beta^n\bar{y}-\lambda & -\beta^n\bar{x} \\ \beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad \left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}-\lambda\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\right)+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}=0$$

$$\Rightarrow \quad \left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)-\lambda\left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}\right)$$

$$-\lambda\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)+\lambda^2+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}=0$$

$$\Rightarrow \quad \lambda^2-\left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)\lambda+\beta^{2n}\,\bar{x}\,\bar{y}\,e^{-\delta\tau}e^{-\lambda\tau}$$

$$+\left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0$$

Now we define,

$$\bar{F}(\lambda)=\lambda^2-\left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)\lambda$$

$$+\beta^{2n}\bar{x}\,\bar{y}e^{-\delta\tau}e^{-\lambda\tau}+\left(r^n-\frac{2\bar{x}r^n}{K}-\beta^n\bar{y}\right)\left(-d_0+\beta^n e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0 \qquad \text{... (2.3)}$$

At the equilibrium point $\bar{E}=(\bar{x},\bar{y})=\left(\frac{d_0e^{\delta\tau}}{\beta^n},\frac{r^n}{\beta^n}\left(1-\frac{d_0}{\beta^nK}e^{\delta\tau}\right)\right)$, the equation (2.3) becomes

$$\bar{F}(\lambda)=\lambda^2-\left(r^n-\frac{2d_0e^{\delta\tau}r^n}{\beta^nK}-\beta^n\frac{r^n}{\beta^n}\left(1-\frac{d_0}{\beta^nK}e^{\delta\tau}\right)-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\frac{d_0e^{\delta\tau}}{\beta^n}\right)\lambda$$

$$+\beta^{2n}\frac{d_0e^{\delta\tau}}{\beta^n}\frac{r^n}{\beta^n}\left(1-\frac{d_0}{\beta^nK}e^{\delta\tau}\right)e^{-\delta\tau}e^{-\lambda\tau}$$

$$+\left(r^n-2\frac{d_0e^{\delta\tau}r^n}{\beta^nK}-\beta^n\frac{r^n}{\beta^n}\left(1-\frac{d_0}{\beta^nK}e^{\delta\tau}\right)\right)\left(-d_0+\beta^ne^{-\delta\tau}e^{-\lambda\tau}\frac{d_0e^{\delta\tau}}{\beta^n}\right)=0$$

$$\Rightarrow \quad \bar{F}(\lambda)=\lambda^2-\left(-d_0+d_0e^{-\lambda\tau}-\frac{r^n d_o}{\beta^n K}e^{\delta\tau}\right)\lambda+r^n d_0 e^{-\lambda\tau}-\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}e^{-\lambda\tau}$$

$$-\frac{r^n d_0}{\beta^n K}e^{\delta\tau}\left(d_0e^{-\lambda\tau}-d_0\right)=0$$

$$\Rightarrow \quad \bar{F}(\lambda)=\lambda^2-\left(-d_0+d_0e^{-\lambda\tau}-\frac{r^n d_o}{\beta^n K}e^{\delta\tau}\right)\lambda+r^n d_0 e^{-\lambda\tau}-\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}e^{-\lambda\tau}$$

$$-\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}e^{-\lambda\tau}+\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}=0$$

$$\Rightarrow \quad \bar{F}(\lambda)=\lambda^2-\left(d_0e^{-\lambda\tau}-d_0-\frac{r^n d_o}{\beta^n K}e^{\delta\tau}\right)\lambda+\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}$$

$$+\left(r^n d_0-\frac{2r^n d_0^2}{\beta^n K}e^{\delta\tau}\right)e^{-\lambda\tau}=0 \qquad \ldots(2.4)$$

If $\tau=0$, then (2.4) becomes

$$\lambda^2-\left(-\frac{r^n d_o}{\beta^n K}\right)\lambda+\frac{r^n d_0^2}{\beta^n K}+\left(r^n d_0-\frac{2r^n d_0^2}{\beta^n K}\right)=0$$

$$\Rightarrow \quad \lambda^2+\frac{r^n d_o}{\beta^n K}\lambda+\left(r^n d_0-\frac{r^n d_0^2}{\beta^n K}\right)=0$$, which is a quadratic equation in λ.

$$\therefore \quad \lambda=\frac{-\frac{r^n d_o}{\beta^n K}\pm\sqrt{\left(\frac{r^n d_0}{\beta^n K}\right)^2-4\left(r^n d_0-\frac{r^n d_0^2}{\beta^n K}\right)}}{2}$$

$$\Rightarrow \quad \lambda=-\frac{1}{2}\frac{r^n d_o}{\beta^n K}\pm\frac{1}{2}\sqrt{\left(\frac{r^n d_0}{\beta^n K}\right)^2-4r^n d_0\left(1-\frac{d_0}{\beta^n K}\right)}$$

We consider

$$\lambda_1=-\frac{1}{2}\frac{r^n d_0}{\beta^n K}-\frac{1}{2}\sqrt{\left(\frac{r^n d_0}{\beta^n K}\right)^2-4r^n d_0\left(1-\frac{d_0}{\beta^n K}\right)}$$

and $\lambda_2 = -\frac{1}{2}\frac{r^n d_0}{\beta^n K} + \frac{1}{2}\sqrt{\left(\frac{r^n d_0}{\beta^n K}\right)^2 - 4r^n d_0\left(1 - \frac{d_0}{\beta^n K}\right)}$

Here the real part of λ_1 is negative, so the stability depends on another eigen value λ_2.

Theorem 2.2 :

If $\tau = 0$, then $\overline{E}$ is locally asymptotically stable if $d_0 < \beta^n K$ and $\overline{E}$ does not exists if $d_0 > \beta^n K$.

Proof :

In case of positive delay i.e, $\tau > 0$, the characteristic equation for the linearized equation around the point

$\overline{E}\left(\frac{d_0 e^{\delta\tau}}{\beta^n}, \frac{r^n}{\beta^n}\left(1 - \frac{d_0}{\beta^n K}e^{\delta\tau}\right)\right)$ is given by

$$\overline{P}(\lambda) + \overline{Q}(\lambda)e^{-\lambda\tau} = 0, \qquad \ldots (2.5)$$

where $\overline{P}(\lambda) = \lambda^2 + p_1\lambda + p_2$ and $\overline{Q}(\lambda) = q_1\lambda + q_2$

Here $p_1 = d_0 + \frac{r^n d_0}{\beta^n K}e^{\delta\tau} > 0,$

$p_2 = \frac{r^n d_0^2}{\beta^n K}e^{\delta\tau} > 0,$

$q_1 = -d_0 < 0,$

$q_2 = r^n d_0 - \frac{2r^n d_0^2}{\beta^n K}e^{\delta\tau}$

If $\tau > 0$, let $\lambda = iw,\ w > 0$ be a purely imaginary root of (2.5)

i.e, $\overline{P}(\lambda) + \overline{Q}(\lambda)e^{-\lambda\tau} = 0$

$$\Rightarrow \quad \lambda^2 - \left(d_0 e^{-\lambda\tau} - d_0 - \frac{r^n d_0}{\beta^n K}e^{\delta\tau}\right)\lambda + \frac{r^n d_0^2}{\beta^n K}e^{\delta\tau} + \left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K}e^{\delta\tau}\right)e^{-\lambda\tau} = 0$$

i.e, $\overline{F}(\lambda) = \lambda^2 - \left(d_0 e^{-\lambda\tau} - d_0 - \frac{r^n d_0}{\beta^n K}e^{\delta\tau}\right)\lambda + \frac{r^n d_0^2}{\beta^n K}e^{\delta\tau} + \left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K}e^{\delta\tau}\right)e^{-\lambda\tau} = 0$

Now substituting $\lambda = iw$ in $\overline{F}(\lambda)$ we get ,

$$\bar{F}(iw) = (iw)^2 - iw\left(d_0 e^{-iw\tau} - d_0 - \frac{r^n d_0}{\beta^n K} e^{\delta\tau}\right) + \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau} + \left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right) e^{-iw\tau} = 0$$

$$\Rightarrow \quad -w^2 - iwd_0(\cos w\tau - i\sin w\tau) + iwd_0 + \frac{iwr^n d_0}{\beta^n K} e^{\delta\tau} + \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau}$$

$$+\left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right)(\cos w\tau - i\sin w\tau) = 0$$

$$\Rightarrow \quad \left(-w^2 - wd_0 \sin w\tau + \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau} + r^n d_0 \cos w\tau - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau} \cos w\tau\right)$$

$$+i\left(-wd_0 \cos w\tau + wd_0 + \frac{wr^n d_0}{\beta^n K} e^{\delta\tau} - r^n d_0 \sin w\tau + \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau} \sin w\tau\right) = 0$$

Equating the real and imaginary parts we get the system of transcendental equations

$$-w^2 - wd_0 \sin w\tau + \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau} + r^n d_0 \cos w\tau - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau} \cos w\tau = 0,$$

$$-wd_0 \cos w\tau + wd_0 + \frac{wr^n d_0}{\beta^n K} e^{\delta\tau} - r^n d_0 \sin w\tau + \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau} \sin w\tau = 0.$$

$$\Rightarrow \quad \left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right) \cos w\tau - wd_0 \sin w\tau = w^2 - \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau} \quad \ldots(2.6)$$

and
$$\left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right) \sin w\tau + wd_0 \cos w\tau = wd_0 + \frac{wr^n d_0}{\beta^n K} e^{\delta\tau} \quad \ldots(2.7)$$

Squaring and adding (2.6) and (2.7), we get

$$\left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right)^2 + w^2 d_0^2 - 2\left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right) wd_0 \sin w\tau \cos w\tau$$

$$+2\left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right) wd_0 \sin w\tau \cos w\tau$$

$$= \left(w^2 - \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau}\right)^2 + \left(wd_0 + \frac{wr^n d_0}{\beta^n K} e^{\delta\tau}\right)^2$$

$$\Rightarrow \quad \left(r^n d_0 - \frac{2r^n d_0^2}{\beta^n K} e^{\delta\tau}\right)^2 + w^2 d_0^2 = \left(w^2 - \frac{r^n d_0^2}{\beta^n K} e^{\delta\tau}\right)^2 + \left(wd_0 + \frac{wr^n d_0}{\beta^n K} e^{\delta\tau}\right)^2$$

$$\Rightarrow \quad w^4 + \frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau} w^2 + \left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right) = 0,$$

which is a quadratic equation in w^2

$$\therefore \quad w^2 = \frac{-\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau} \pm \sqrt{\left(\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau}\right)^2 - 4\left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right)}}{2}$$

$$\Rightarrow \quad w^2 = \frac{-r^{2n} d_0^2}{2\beta^{2n} K^2} e^{2\delta\tau} \pm \frac{1}{2}\sqrt{\left(\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau}\right)^2 - 4\left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right)}$$

Since, $w^2 > 0$, we have

$$\frac{-r^{2n} d_0^2}{2\beta^{2n} K^2} e^{2\delta\tau} + \frac{1}{2}\sqrt{\left(\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau}\right)^2 - 4\left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right)} > 0$$

$$\Rightarrow \quad \left(\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau}\right)^2 - 4\left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right) > \left(\frac{r^{2n} d_0^2}{\beta^{2n} K^2} e^{2\delta\tau}\right)^2$$

$$\Rightarrow \quad -4\left(\frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau}\right) > 0$$

$$\Rightarrow \quad \frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} - r^{2n} d_0^2 - \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau} < 0$$

$$\Rightarrow \quad \frac{3r^{2n} d_0^4}{\beta^{2n} K^2} e^{2\delta\tau} - \frac{4r^{2n} d_0^3}{\beta^n K} e^{\delta\tau} + r^{2n} d_0^2 > 0$$

$$\Rightarrow \quad \frac{r^{2n} d_0^2}{\beta^{2n} K^2}\left(3e^{2\delta\tau} d_0^2 - 4\beta^n K d_0 e^{\delta\tau} + \beta^{2n} K^2\right) > 0$$

$$\Rightarrow \quad 3e^{2\delta\tau} d_0^2 - 4\beta^n K d_0 e^{\delta\tau} + \beta^{2n} K^2 > 0$$

$$\Rightarrow \quad e^{2\delta\tau} d_0^2 - \frac{4}{3}\beta^n K d_0 e^{\delta\tau} + \frac{1}{3}\beta^{2n} K^2 > 0$$

$$\Rightarrow \quad e^{2\delta\tau} - \frac{4\beta^n K e^{\delta\tau}}{3d_0} + \frac{\beta^{2n} K^2}{3d_0^2} > 0$$

$$\Rightarrow \quad \left(e^{\delta\tau}\right)^2 - 2e^{\delta\tau}\left(\frac{2\beta^n K}{3d_0}\right) + \left(\frac{2\beta^n K}{3d_0}\right)^2 - \left(\frac{2\beta^n K}{3d_0}\right)^2 + \frac{\beta^{2n}K^2}{3d_0^2} > 0$$

$$\Rightarrow \quad \left(e^{\delta\tau} - \frac{2\beta^n K}{3d_0}\right)^2 + \frac{\beta^{2n}K^2}{3d_0^2} - \frac{4\beta^{2n}K^2}{9d_0^2} > 0$$

$$\Rightarrow \quad \left(e^{\delta\tau} - \frac{2\beta^n K}{3d_0}\right)^2 - \left(\frac{\beta^n K}{3d_0}\right)^2 > 0$$

$$\Rightarrow \quad \left(e^{\delta\tau} - \frac{2\beta^n K}{3d_0}\right)^2 > \left(\frac{\beta^n K}{3d_0}\right)^2 \qquad \ldots(2.8)$$

if $\quad e^{\delta\tau} - \frac{2\beta^n K}{3d_0} < -\frac{\beta^n K}{3d_0}$ or $\quad e^{\delta\tau} - \frac{2\beta^n K}{3d_0} > \frac{\beta^n K}{3d_0}$

if $\quad e^{\delta\tau} < \frac{\beta^n K}{3d_0}$ or $\quad e^{\delta\tau} > \frac{3\beta^n K}{3d_0}$

if $\quad e^{\delta\tau} < \frac{\beta^n K}{3d_0}$ or $\quad e^{\delta\tau} > \frac{\beta^n K}{d_0}$

Now, $e^{\delta\tau} < \frac{\beta^n K}{3d_0}$

$$\Rightarrow \quad \frac{\beta^n K}{3d_0} > e^{\delta\tau} > 1, \text{ when } \delta > 0, \tau > 0$$

$$\Rightarrow \quad \frac{\beta^n K}{3d_0} > 1$$

$$\Rightarrow \quad \beta^n K > 3d_0$$

$$\Rightarrow \quad d_0 < \frac{1}{3}\beta^n K < \beta^n K$$

$$\Rightarrow \quad d_0 < \beta^n K$$

Again, $e^{\delta\tau} > \frac{\beta^n K}{d_0}$

$$\Rightarrow \quad \frac{\beta^n K}{d_0} < e^{\delta\tau} < 1, \text{ when } \delta < 0, \tau > 0$$

$$\Rightarrow \quad \beta^n K < d_0 \text{ when } \delta < 0, \tau > 0$$

$\Rightarrow \quad d_0 > \beta^n K$ when $\delta < 0, \tau > 0$

But δ is not negative, so $d_0 > \beta^n K$ is not possible.

Hopf-bifurcation

We will now show that

$$\left[\frac{d}{d\tau}(\text{Re}\,\lambda)\right]_{\tau=\tau_0} > 0$$

This will signify that there exists at least one eigen value with positive real part for $\tau > \tau_0$. Also the conditions for Hopf bifurcation [6] are then satisfied giving the required periodic solution.

We first look for purely imaginary roots of $\lambda = iw_0$ of (2.5)

From equation (2.5)

$$\overline{P}(\lambda) = -\overline{Q}(\lambda)e^{-\lambda\tau}$$

$$\Rightarrow \quad \left|\overline{P}(iw_0)\right| = \left|-\overline{Q}(iw_0)e^{-iw_0\tau}\right|$$

$$\Rightarrow \quad \left|\overline{P}(iw_0)\right| = \left|\overline{Q}(iw_0)\right|\left|e^{-iw_0\tau}\right|$$

$$\Rightarrow \quad \left|\overline{P}(iw_0)\right| = \left|\overline{Q}(iw_0)\right|\left|\cos(w_0\tau) - i\sin(w_0\tau\right|$$

$$\Rightarrow \quad \left|\overline{P}(iw_0)\right| = \left|\overline{Q}(iw_0)\right|$$

and this determines a set of possible values of w_0.

Our, aim is to determine the direction of motion of δ as τ is varied.

i.e, we determine

$$sign\left[\frac{d}{d\tau}(\text{Re}\,\lambda)\right]_{\lambda=iw_0} = sign\left[\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=iw_0}.$$

We have from (2.5),

$$\overline{P}(\lambda) + \overline{Q}(\lambda)e^{-\lambda\tau} = 0$$

$$\Rightarrow \quad \left(\lambda^2 + p_1\lambda + p_2\right) + \left(q_1\lambda + q_2\right)e^{-\lambda\tau} = 0 \qquad \ldots(2.9)$$

To find $\dfrac{d\lambda}{d\tau}$,

differentiating (2.9) w.r.t τ, we get

$$\frac{d}{d\tau}\left[\left(\lambda^2+p_1\lambda+p_2\right)+\left(q_1\lambda+q_2\right)e^{-\lambda\tau}\right]=0$$

$$\Rightarrow \quad (2\lambda+p_1)\frac{d\lambda}{d\tau}+e^{-\lambda\tau}\frac{d}{d\tau}(q_1\lambda+q_2)+(q_1\lambda+q_2)\frac{d}{d\tau}\left(e^{-\lambda\tau}\right)=0$$

$$\Rightarrow \quad (2\lambda+p_1)\frac{d\lambda}{d\tau}+e^{-\lambda\tau}q_1\frac{d\lambda}{d\tau}+(q_1\lambda+q_2)e^{-\lambda\tau}\left(-\lambda-\tau\frac{d\lambda}{d\tau}\right)=0$$

$$\Rightarrow \quad \left[(2\lambda+p_1)+e^{-\lambda\tau}q_1-\tau e^{-\lambda\tau}(q_1\lambda+q_2)\right]\frac{d\lambda}{d\tau}=\lambda(q_1\lambda+q_2)e^{-\lambda\tau}$$

$$\Rightarrow \quad \frac{d\tau}{d\lambda}=\frac{\left[(2\lambda+p_1)+q_1e^{-\lambda\tau}-\tau e^{-\lambda\tau}(q_1\lambda+q_2)\right]}{\lambda(q_1\lambda+q_2)e^{-\lambda\tau}}$$

$$\Rightarrow \quad \left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{2\lambda+p_1}{\lambda(q_1\lambda+q_2)e^{-\lambda\tau}}+\frac{q_1e^{-\lambda\tau}}{\lambda(q_1\lambda+q_2)e^{-\lambda\tau}}-\frac{\tau}{\lambda}$$

$$\Rightarrow \quad \left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{2\lambda+p_1}{\lambda(q_1\lambda+q_2)\,e^{-\lambda\tau}}+\frac{q_1}{\lambda(q_1\lambda+q_2)}-\frac{\tau}{\lambda}$$

Therefore,

$$sign\left[\frac{d}{d\tau}(\operatorname{Re}\lambda)\right]_{\lambda=iw_0}$$

$$=sign\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=iw_0}$$

$$=sign\left[\operatorname{Re}\left(\frac{2\lambda+p_1}{-\lambda\left(\lambda^2+p_1\lambda+p_2\right)}+\frac{q_1}{\lambda(q_1\lambda+q_2)}-\frac{\tau}{\lambda}\right)\right]_{\lambda=iw_0} \quad \text{(using (2.9))}$$

$$=sign\left[\operatorname{Re}\left(\frac{2iw_0+p_1}{-iw_0\left(i^2w_0^2+p_1iw_0+p_2\right)}+\frac{q_1}{iw_0(q_1iw_0+q_2)}-\frac{\tau}{iw_0}\right)\right]$$

$$=sign\left[\operatorname{Re}\left(\frac{2iw_0+p_1}{p_1w_0^2+i\left(w_0^3-p_2w_0\right)}+\frac{q_1}{-q_1w_0^2+iq_2w_0}-\frac{\tau}{iw_0}\right)\right]$$

$$=sign\left[\operatorname{Re}\left(\frac{(2iw_0+p_1)\left(p_1w_0^2-i\left(w_0^3-p_2w_0\right)\right)}{\left(p_1w_0^2\right)^2+\left(w_0^3-p_2w_0\right)^2}+\frac{q_1\left(-q_1w_0^2-iq_2w_0\right)}{\left(-q_1w_0^2\right)^2+(q_2w_0)^2}+\frac{i\tau}{w_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{(2iw_0+p_1)(p_1w_0^2-iw_0^3+ip_2w_0)}{p_1^2w_0^4+(w_0^3-p_2w_0)^2}+\frac{q_1(-q_1w_0^2-iq_2w_0)}{q_1^2w_0^4+q_2^2w_0^2}+\frac{i\tau}{w_0}\right)\right]$$

$$= sign\left[\frac{2w_0(w_0^3-p_2w_0)+p_1^2w_0^2}{p_1^2w_0^4+(w_0^3-p_2w_0)^2}-\frac{q_1^2w_0^2}{q_1^2w_0^4+q_2^2w_0^2}\right]$$

$$= sign\left[\frac{\left(2w_0^4-2p_2w_0^2+p_1^2w_0^2\right)\left(q_1^2w_0^4+q_2^2w_0^2\right)-q_1^2w_0^2\left(p_1^2w_0^4+w_0^6+p_2^2w_0^2-2p_2w_0^4\right)}{\left(p_1^2w_0^4+\left(w_0^3-p_2w_0\right)^2\right)\left(q_1^2w_0^4+q_2^2w_0^2\right)}\right]$$

$$= sign\left[\frac{w_0^8q_1^2+2w_0^6q_2^2-2p_2q_2^2w_0^4+p_1^2q_1^2w_0^4-p_2^2q_1^2w_0^4}{\left(p_1^2w_0^4+\left(w_0^3-p_2w_0\right)^2\right)\ w_0^2\left(q_1^2w_0^2+q_2^2\right)}\right]$$

$$= sign\left[\frac{w_0^6q_1^2+2w_0^4q_2^2-2p_2q_2^2w_0^2+p_1^2q_2^2w_0^2-p_2^2q_1^2w_0^2}{\left(p_1^2w_0^4+\left(w_0^3-p_2w_0\right)^2\right)\left(q_1^2w_0^2+q_2^2\right)}\right]$$

$$= sign\left[\frac{w_0^6q_1^2+2w_0^2q_2^2\left(w_0^2-p_2\right)+w_0^2\left(p_1^2q_2^2-p_2^2q_1^2\right)}{\left(p_1^2w_0^4+\left(w_0^3-p_2w_0\right)^2\right)\left(q_1^2w_0^2+q_2^2\right)}\right]$$

$=1>0$, if and only if $w_0^2-p_2>0$ and $p_1^2q_2^2-p_2^2q_1^2>0$.

Therefore, the transversality condition will hold and hence Hopf-bifurcation will occur at

$w=w_0, \tau=\tau_0$

i.e, $\left[\frac{d}{d\tau}(\mathrm{Re}\,\lambda)\right]_{w=w_0,\tau=\tau_0}>0$

if and only if $w_0^2-p_2>0$ and $p_1^2q_2^2-p_2^2q_1^2>0$.

$$w_0^2-p_2>0$$

$$\Rightarrow \quad w_0^2>p_2$$

$$\Rightarrow \quad w_0^2>\frac{r^nd_0^2}{\beta^nK}e^{\delta\tau} \qquad \ldots (2.10)$$

and $\quad p_1^2q_2^2-p_2^2q_1^2>0$

$$\Rightarrow \quad \left(d_0+\frac{r^n d_0}{\beta^n K}e^{\delta\tau}\right)^2\left(r^n d_0-\frac{2r^n d_0^2}{\beta^n K}e^{\delta\tau}\right)^2-d_0^2\left(\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau}\right)^2>0$$

$$\Rightarrow \quad d_0^4\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)^2\left(r^n-\frac{2r^n d_0}{\beta^n K}e^{\delta\tau}\right)^2-d_0^6\left(\frac{r^n}{\beta^n K}e^{\delta\tau}\right)^2>0$$

$$\Rightarrow \quad d_0^4\left(r^n\right)^2\left[\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)^2\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)^2-d_0^2\left(\frac{e^{\delta\tau}}{\beta^n K}\right)^2\right]>0$$

$$\Rightarrow \quad \left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)^2\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)^2-d_0^2\left(\frac{e^{\delta\tau}}{\beta^n K}\right)^2>0$$

$$\Rightarrow \quad \left[\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)\right]^2>\left(\frac{d_0 e^{\delta\tau}}{\beta^n K}\right)^2$$

if $\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)<-\left(\frac{d_0 e^{\delta\tau}}{\beta^n K}\right)$ or $\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)>\left(\frac{d_0 e^{\delta\tau}}{\beta^n K}\right)$

Conclusion :

In this paper, a mathematical model has been proposed and analyzed to study the dynamics of a predator-prey system due to the time lags for the conversion of biomass and considering different growth functions of prey. The model has been analyzed in two sections: first when growth function of prey is monotonic and second when growth function of prey is logistic. Attempt have also been made to understand the effect of gestation delay on dynamical behavior of predator-prey system. Linear stability analysis reveals that for the monotonic growth rate of prey, in the absence of delay the co-existence equilibrium is a centre. But for the logistic growth function of prey, it is locally asymptotically stable if $d_0<\beta^n K$ and does not exist if $d_0>\beta^n K$. For maintaining co-existence between the predator-prey interaction, balance growth rate of prey and carrying capacity of an environment is necessary. Also we observed that for the monotonic growth rate of prey, in absence of delay , Hopf -bifurcation is not possible but in case of positive delay Hop-f bifurcations is possible without any condition and there is a periodic solution, which is the case of Hopf-bifurcation. In case of logistic growth of prey, Hopf-bifurcation is possible under the condition

$$w_0^2>\frac{r^n d_0^2}{\beta^n K}e^{\delta\tau} \text{ and } \left[\left(1+\frac{r^n}{\beta^n K}e^{\delta\tau}\right)\left(1-\frac{2d_0}{\beta^n K}e^{\delta\tau}\right)\right]^2>\left(\frac{d_0 e^{\delta\tau}}{\beta^n K}\right)^2$$

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