MAGIC AND PARTIALLY MAGIC PYRAMIDAL GRAPHS

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Abstract:
Let G = (V, E) be a graph with p vertices and q edges. The graph G is said to be a Magic Pyramidal graph if it admits both Vertex Magic Pyramidal labeling and Edge Magic Pyramidal labeling. In this paper we investigate graphs that do not satisfy Vertex Magic Pyramidal labeling but satisfy Edge Magic Pyramidal labeling and categorize those graphs as Partially Magic Pyramidal graphs.

Key words: Pyramidal, Magic, Vertex Magic, Edge Magic

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I. INTRODUCTION

Graph labeling is an assignment of labels to the vertices or edges or to both the vertices and edges of a graph subject to certain conditions. Magic labelings have their origin from magic squares and it was first introduced by Sedlacek. The concept of Vertex Magic Pyramidal and Edge Magic Pyramidal labelings are introduced with Pyramidal numbers. In a Vertex magic pyramidal labeling the weight of a vertex is the sum of the vertex label and the labels of the edges incident with that vertex. In Vertex magic pyramidal labeling the weight of each vertex is a constant and the constant must be a pyramidal number. For a particular graph there are many vertex magic constants. Analogous to vertex magic pyramidal labeling edge magic pyramidal labeling is defined in such a way that at each edge the sum of that edge label and the labels of the vertices incident with that edge is a constant and the constant must be a pyramidal number. In this paper we investigate graphs which admit both Vertex Magic Pyramidal labeling and Edge Magic Pyramidal labeling and also find Partially Magic Pyramidal graphs.

II. MAGIC PYRAMIDAL GRAPHS

Definition 2.1: A Triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n and is got from the formula $T_n = \frac{n(n+1)}{2}$. The sum of Consecutive triangular numbers is known as tetrahedral numbers. (i.e.) 1, 4, 10, 20, 35...

Definition 2.2: The Pyramidal numbers or Square Pyramidal numbers are the sums of consecutive pairs of tetrahedral numbers. They are 1, 5, 14, 30, 55.... They can be calculated from the formula $p_n = \frac{n(n+1)(2n+1)}{6}$. The notation p_i is used for each Pyramidal number where i = 1,2,3....

Definition 2.3. The Vertex Magic Pyramidal labeling of a Graph G=(V,E) is defined as a one-to-one function f (we call Vertex Magic Pyramidal function) from V(G)∪E(G) onto the integers$\{1,2,3,\ldots,p_q\}$ with the property that there is a constant λ_f such that f(u) +$\sum f(uv)$ = λ_f where the sum runs over all vertices v adjacent to u and uv is the edge joining the vertices u and v and the constant λ_f must be a Pyramidal number. Here p_q denotes the q^{th} Pyramidal number. The constant λ_f is called the Vertex Magic constant of the given graph. A graph G which satisfies the above labeling is called a vertex magic pyramidal graph.

Definition 2.4. The Edge Magic Pyramidal labeling of a Graph G=(V,E) is defined as a one-to-one function f (we call Edge Magic Pyramidal function) from V(G)∪E(G) onto the integers$\{1,2,3,\ldots,p_q\}$ with the property that there is a constant μ_f such that f(u) + f(v) +f(uv) = μ_f where uv ∈ $E(G)$ and the constant μ_f must be a Pyramidal number. Here p_q denotes the q^{th} Pyramidal number. The constant μ_f is called the Edge Magic constant of the given graph. A graph G which satisfies the above labeling is called an edge magic pyramidal graph.

Remark 2.5. For a graph G, there can be many Vertex Magic Pyramidal functions and for each function f there is a Vertex Magic constant. Similarly for a graph G, there can be many Edge Magic Pyramidal functions and for each function f there is an Edge Magic constant.

Definition 2.6. A graph Gwhich admits both vertex magic pyramidal labeling and edge magic pyramidal labeling is said to be a Magic Pyramidal graph. In other words, if a graph is both vertex magic pyramidal and edge magic pyramidal then it is termed as a Magic Pyramidal graph.

Theorem 2.7. All Paths P_n are Vertex Magic Pyramidal for n ≥ 4 and the vertex magic constants λ_f range from $\left\lceil\frac{5n-1}{2}\right\rceil \leq \lambda_f \leq p_{n-1}$ for 4 ≤ n ≤ 9 and $\left\lceil\frac{5n+7}{2}\right\rceil \leq \lambda_f \leq p_{n-1}$ for all n ≥ 10 and they are Edge Magic Pyramidal for n ≥ 4 and the edge magic constants μ_f range from $3n \leq \mu_f \leq p_{n-1}$ for all n ≥ 4, where p_{n-1} is the $(n-1)^{th}$ Pyramidal number.

Proof. Case 1. n is odd, n ≥ 5. Let $v_1,v_2,\ldots,v_n$ be the vertices of the Path, e_i, i = 1 to n-1 be the edges of the Path. Define f:V∪ $E \to \{1,2,3,\ldots,p_q\}$ as follows:

$f(v_1) = \lambda_f - 1$ where λ_f is the magic constant.

$$f(v_i) = \begin{cases} f(v_{i-1}) - 2 & for\ 2 \leq i \leq n-1 \\ \lambda_f - (i-1) & for\ i = n \end{cases}$$

$f(e_i) = i$ for $1 \leq i \leq n-1$

Case 2. n is even, n ≥ 4 .

Define $f(v_1) = \lambda_f - 2$ where λ_f is the magic constant.

$$f(v_i) = \begin{cases} f(v_{i-1}) - `1 & \text{for } i = 2 \\ f(v_{i-1}) - 2 & \text{for } 3 \leq i \leq n-1 \\ \lambda_f - i & \text{for } i = n \end{cases}$$

$$f(e_i) = \begin{cases} i+1 & \text{for } i = 1 \\ i-1 & \text{for } i = 2 \\ f(e_{i-2}) + 2 & \text{for } 3 \leq i \leq n-1 \end{cases}$$

By the labeling given in Case 1 and case 2 all Paths P_n are Vertex Magic Pyramidal for n ≥ 4.

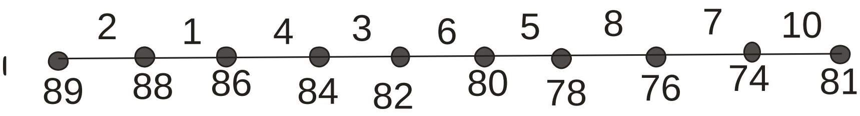
Case 3. Let P_n be such that n ≥ 4. Define f:V∪ $E \to \{1,2,3,\ldots,p_q\}$ as follows:

$f(v_i) = i$ for $1 \leq i \leq n$.

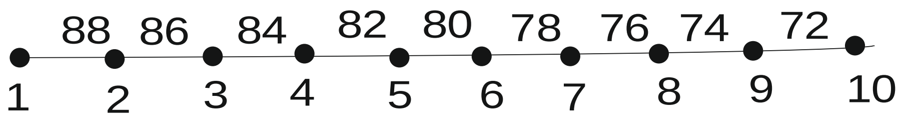
$$f(e_i) = \begin{cases} \mu_f - 3 & for\ i = 1 \\ f(e_{i-1}) - 2 & for\ 2 \leq i \leq n-1 \end{cases}$$

By the labelling in case 3 all Paths P_n are Edge Magic Pyramidal for n ≥ 4. Hence all Paths P_n are Magic Pyramidal graphs for n ≥ 4.

Example:



Vertex Magic Pyramidal labeling of $P_{10}(\lambda_f = 91 = p_6)$.



Edge Magic Pyramidal labeling of P_{10} $(\mu_f = 91 = p_6)$.

Remark 2.8. For P_{10}, in this example we have given λ_f the value p_6 = 91, We have $\left\lceil\frac{5n+7}{2}\right\rceil \le \lambda_f \le p_{n-1}$ and therefore λ_f can take all the pyramidal numbers lying between $\left\lceil\frac{5n+7}{2}\right\rceil$ and $p_{n-1} = p_9$ =285 and hence the possible values of λ_f are 30, 55, 91, 140, 204 and 285. For Edge Magic Pyramidal labelling we have 3n ≤ $\mu_f \le p_{n-1}$ and therefore μ_f can take all the pyramidal numbers lying between 3n = 30 and $p_{n-1} = p_9 =$ 285 and hence the possible values of μ_f are also 30, 55, 91, 140, 204 and 285.

Remark 2.9. All Cycles, Stars, Complete bipartitite graphs, Peterson graph satisfy both vertex magic pyramidal and edge magic pyramidal labeling. Hence they are popularly termed as Magic Pyramidal graphs. All Cycles C_n are Vertex as well as Edge Magic Pyramidal with

$4n+1 \le \lambda_f = \mu_f \le p_{n+1}$ for 3 ≤ n ≤ 7 and $5n+5 < \lambda_f = \mu_f \le p_{n+2}$ ∀ n ≥ 8, Stars with

$\frac{n^2+3n}{2} < \lambda_f \le p_n$, $2n+3 \le \mu_f \le p_n$ for $n \ge 3$, Peterson graph with $p_{m-3} \le \lambda_f \le p_{n+2}$,

$p_{m-3} \le \mu_f \le p_n$.

Remark 2.10. All Complete bipartite graphs $K_{m,n}$, are Vertex Magic Pyramidal with $p_{m+1} \le \lambda_f \le p_{mn}$ for $m \ne n, m > n$ and $p_{n+1} \le \lambda_f \le p_{mn}$ for $m \ne n, n > m$. For $m = n$ λ_f ranges from $p_{m+2} \le \lambda_f \le p_{mn}$ and Edge Magic Pyramidal with $p_{m+2} \le \mu_f \le p_{m^2+i}$ for $m = n$ where i takes the values 0,2,4,6… for each m ranging from 2,3,4…. and for $m \ne n$, μ_f approximately varies from $p_{m+2} \le \mu_f \le p_{mn+m\sim n}$.

III. PARTIALLY MAGIC PYRAMIDAL GRAPHS

Definition 3.1. Graphs that are Edge magic pyramidal but not Vertex magic pyramidal are called as Partially magic Pyramidal graphs.

Lemma 3.2. If a graph G= (V, E) has atleast three Cycles such that d(v) ≥ 6 for some v∈V then G is not Vertex magic pyramidal. Also if G has four or more Cycles such that d(v) ≥ 3 for atleast three vertices then G is not Vertex magic pyramidal.

Proof. Let G = (V,E) be a graph with p vertices and q edges. Assume that G has atleast three Cycles and d(v) ≥ 6 for some v∈V . Since d(v) ≥ 6 there are 6 edges adjacent to the vertex v. Hence including the vertex v, seven labels must be assigned altogether. In such a case a pyramidal number must be partitioned into seven distinct parts. Such a partition is possible. But G has atleast three Cycles. Without loss of generality assume that these three Cycles are of length 3. Therefore satisfying the condition of Vertex magic pyramidal labeling nine distinct labels are required to label the other vertices and edges. In such a case distinct partition of the pyramidal number is not possible. Hence G is not Vertex magic pyramidal. Similar case is true for a graph with four or more Cycles.

Theorem 3.3. All Closed helms obtained from a helm by joining each pendent vertex to form a Cycle are Partially magic pyramidal graphs for n ≥ 4.

Proof. Let G be a Closed helm obtained from a helm by joining each pendent vertex to form a Cycle has 2n+1 vertices and 4n edges where n ≥ 4. Let v_0 be the central vertex. Let v_i, $1 \leq i \leq n$ be the vertices of the Cycle C_n in the Helm H_n. Let w_i where $1 \leq i \leq n$ be the pendent vertices of the Helm joined together to form a Cycle.

Define f: V(G)∪E(G).→ $\{1,2,3,\ldots,p_q\}$ as follows:

f(v_0) = 2n

$f(v_i) =$ i for $1 \leq i \leq n$

$f(w_1) = 2f(v_0)$

f(w_i) = $f(w_{i-1}) - 1$ for $2 \leq i \leq n$

$f(e_i) = \mu_f - \sum f(v)$ for i = 1 to q where $\sum f(v)$ denote the sum of the labels of the vertices incident with the edge e_i and μ_f is the Edge magic constant. By the above labeling all Closed helms are Edge magic pyramidal but not Vertex magic pyramidal by lemma 3.2. Hence they are Partially magic pyramidal graphs.

Theorem 3.4. All Triangular Crocodiles with atleast four Cycles are partially magic Pyramidal graphs.

Proof. Let G be a Triangular Crocodile with atleast four Cycles obtained from a Path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to new vertices w_i for i = 1,2,…n−1and z_i for i = 1,2,…n−1. Let p denote the number of vertices and q denote the number of edges in the graph. Define f: V(G)∪E(G).→ $\{1,2,3,\ldots,p_q\}$ as follows:

f(v_i) = i, for $1 \leq i \leq n$

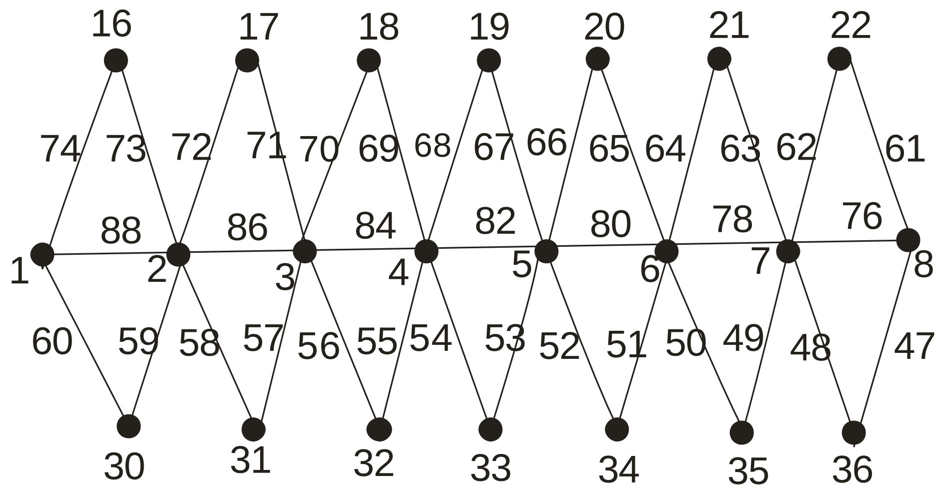
f(w_1) = 2f(v_n)

f(w_i) = f(w_{i-1}) + 1 for $2 \leq i \leq n-1$

f(z_1) = f(w_{n-1}) + n

f(z_i) = f(z_{i-1}) + 1 for $2 \leq i \leq n-1$

f(e_i) = $\mu_f - \sum f(v)$ for i = 1 to q where $\sum f(v)$ denote the sum of the labels of the vertices incident with the edge e_i and μ_f is the Edge magic constant. By the above labeling all Triangular Crocodiles with atleast four Cycles are Edge magic pyramidal but not Vertex magic pyramidal by lemma 3.2.. Hence they are Partially magic pyramidal graphs.



Partially Magic Pyramidal labeling of a Triangular Crocodile (μ_f = 91).

Theorem 3.5. All book graphs B_m, m ≥ 4 are Partially magic pyramidal graphs.

Proof. The Book graph $G = B_m$ is the graph $S_m \times P_2$ where S_m is the Star with m+1 vertices. It resembles a book. Let G = B_m be such that m ≥ 4. Let v_1, v_2 be the vertices of the Path P_2. Let $v_{1,i}$ where $1 \le i \le m$ be the vertices of the Star incident with v_1. . Let $v_{2,i}$ where $1 \le i \le m$ be the vertices of the Star incident with v_2.

Define f: V(G)∪E(G).→ $\{1,2,3,\ldots,p_q\}$ as follows:

f(v_i) = i for i = 1,2

$f(v_{2,1}) = 3f(v_2)$

$f(v_{2,i}) = f(v_{2,i-1}) + 1$ for $2 \le i \le m$

$f(v_{1,1}) = 3f(v_{2,m})$

$f(v_{1,i}) = f(v_{1,i-1}) + 1$ for $2 \le i \le m$

$f(e_i) = \mu_f - \sum f(v)$ for i = 1 to q where $\sum f(v)$ denote the sum of the labels of the vertices incident with the edge e_i and μ_f is the Edge magic constant. By the above labeling all Book graphs are Partially magic pyramidal graphs.

Theorem 3.6. The Sunflower graphs are Partially magic Pyramidal graphs.

Proof. Let G be a Sunflower graph obtained by taking a Wheel with the central vertex v_0 and the Cycle $v_1, v_2,\ldots,v_n, v_1$ and additional vertices $w_1, w_2,\ldots,w_n$ where w_i is joined by edges to v_i, v_{i+1} where (i+1) is taken modulo n. It has (2n+1) vertices and 5n edges.

Define f: V(G)∪E(G).→ $\{1,2,3,\ldots,p_q\}$ as follows:

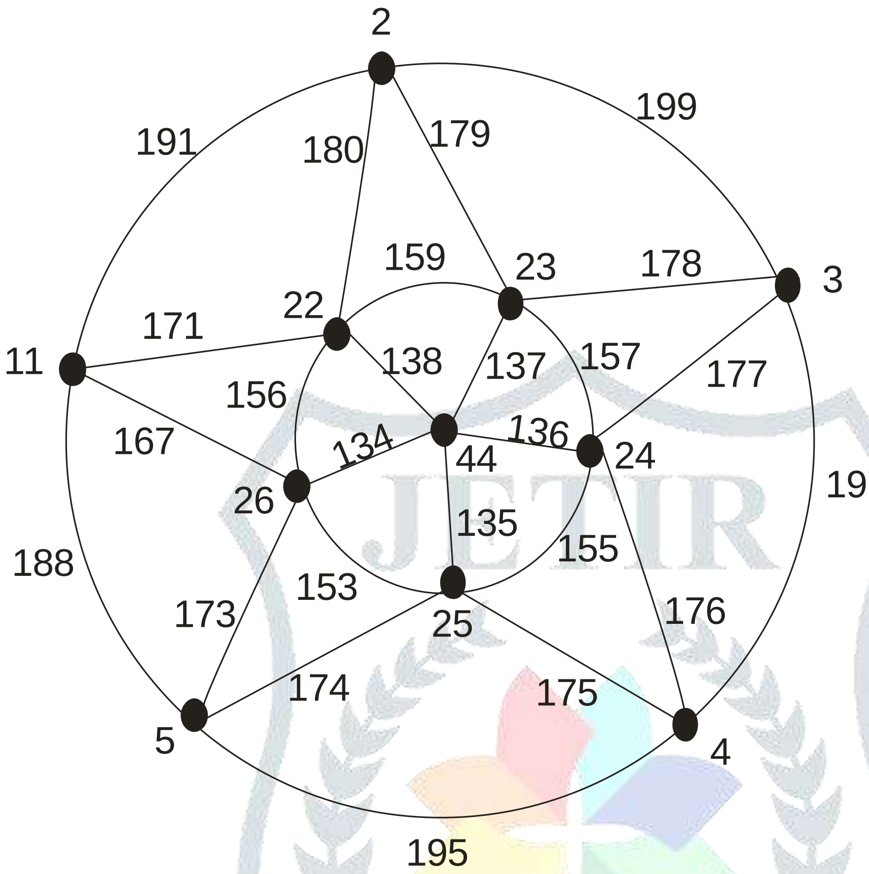
$$f(w_i) = \begin{cases} i+1 & for\ 1 \le i \le n-1 \\ 2i+1 & for\ i = n \end{cases}$$

$f(v_1) = 2f(w_n)$

$f(v_i) = f(v_{i-1}) + 1$ for $2 \le i \le n-1$

$f(v_0) = 2f(v_1)$

$f(e_i) = \mu_f - \sum f(v)$ for i = 1 to q where $\sum f(v)$ denote the sum of the labels of the vertices incident with the edge e_i and μ_f is the Edge magic constant. By the above labeling the Sunflower graphs are Edge magic pyramidal but not Vertex magic pyramidal. Hence all Sunflower graphs are Partially magic pyramidal graphs.



Partially Magic Pyramidal labeling of a Sunflower graph (μ_f = 204).

IV. CONCLUSION

Whenever a graph has atleast three cycles with the degree of a vertex sufficiently large exceeding the number of cycles then the graph fails to be a vertex magic pyramidal graph, as the pyramidal number cannot be partitioned into as many distinct parts as the degree of a vertex. Since every edge in a graph is incident with two vertices, most of the graphs satisfy the condition of Edge magic pyramidal labeling. Hence other classes of Partially magic pyramidal graphs may also be investigated and the range of the magic constants can be determined for those graphs.

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