PERTURBATION ANALYSIS OF MHD CASSON FLOW OF A VISCO-ELASTIC FLUID PAST AN INFINITE VERTICAL POROUS PLATE WITH THERMAL RADIATION AND CHEMICAL REACTION EFFECTS

1R. VIJAYARAGAVAN, 2S. KARThIKEYAN, 3A. SUMITHRA
1Associate Professor, 2Research Scholar, 3Research Scholar
1Department of Mathematics,
1Thiruvalluvar University, Vellore, India.

Abstract: The present study, we considered the effects of unsteady MHD non-Newtonian flow of a Visco-Elastic fluid past an infinite vertical porous plate and the effects of thermal radiation and chemical reaction along with heat and mass transfer are reported. The governing equations are transformed into nonlinear ordinary differential equations using suitable transformation and then solved analytically by using perturbation technique. The velocity, temperature and concentration are presented graphically with help of various physical parameters.

Index Terms: MHD, Casson fluid, Visco-Elastic fluid, Thermal radiation, Chemical reaction.

1INTRODUCTION

Numerical applications of Visco-elastic fluid in several manufacturing processes have led scientists to investigate Visco-elastic flow on the boundary layer. The study of viscoelastic fluid flowing over a vertical surface immersed in porous media in presence of magnetic field has attracted the researchers because of its applications in geophysics, astrophysics, geo-hydrology, chemical engineering, biological system, soil physics and filtration of solid from liquids.


© 2018 JETIR November 2018, Volume 5, Issue 11 www.jetir.org (ISSN-2349-5162)

In the present study, the effects of unsteady MHD non-Newtonian flow of a Visco-Elastic fluid past an infinite vertical porous plate and the effects of thermal radiation and chemical reaction along with heat and mass transfer are reported. The governing equations are transformed into nonlinear ordinary differential equations using suitable transformation and then solved analytically by using perturbation technique. The velocity, temperature and concentration are presented graphically with help of various physical parameters.

II. MATHEMATICAL FORMULATION

Consider unsteady two-dimensional flow of a laminar, viscoelastic, electrically conducting and heat absorbing fluid past a semi-infinite vertical permeable moving plate embedded in a uniform porous medium and subjected to a uniform transverse magnetic field in the presence of thermal and concentration buoyancy effects. It is assumed that there is no applied voltage which implies the absence of an electrical field. The fluid properties are assumed to be constant except that the influence of density variation with temperature. The concentration of diffusing species is very small in comparison to other chemical species, the concentration of species far from the wall $C_{\infty}$, is infinitesimally small and hence the Soret and Dufour effects are neglected. The chemical reaction takes place in the flow and all thermo physical properties which are assumed to be constant on the linear momentum equation is approximated according to the Bossiness approximation. Due to the semi-infinite plane surface assumption, the flow variables are functions of $y'$ and the time $t'$ only.

Under these assumptions, the equations that the physical situation are given by

$$\frac{\partial v'}{\partial y'} = 0$$

\section*{MOMENTUM}

$$\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u'}{\partial y'^2} - \left( \frac{\sigma \beta_0^2}{\rho} + \frac{v'}{k'} \right) u' + g \beta \left( T' - T_\infty \right) + g \beta_c \left( C' - C_\infty \right) - k_0 \left( \frac{\partial^2 u'}{\partial t' \partial y'^2} + v' \frac{\partial^2 u'}{\partial y'^3} \right)$$

\section*{TEMPERATURE}

$$\frac{\partial T'}{\partial t'} + v' \frac{\partial T'}{\partial y'} = \frac{k}{\rho c_p} \frac{\partial^2 T'}{\partial y'^2} + Q_0 \left( T' - T_\infty \right) + Q_c \left( C' - C_\infty \right) - \frac{1}{\rho c_p} \frac{\partial q_y}{\partial y'}$$
CONCENTRATION

\[
\frac{\partial C'}{\partial t'} + \mathbf{v'} \cdot \frac{\partial C'}{\partial y'} = D \frac{\partial^2 C'}{\partial y'^2} - K_1 (C' - C_\infty)
\]  

(4)

Where \( \mathbf{x}' \) is the dimensional distances, \( \mathbf{y}' \) is perpendicular to the plate and \( t' \) is the dimensional time. \( \mathbf{u}' \) and \( \mathbf{v}' \) are the components of dimensional velocities along \( \mathbf{x} \) and \( \mathbf{y} \) directions, respectively. \( T' \) is the dimensional temperature, \( C' \) is the dimensional concentration, \( C_w \) and \( T_w \) are the concentration and temperature at the wall, respectively. \( C_\infty \) and \( T_\infty \) are the free stream dimensional concentration and temperature, respectively. \( \rho \) is the fluid density, \( \nu \) is the kinematic viscosity, \( c_p \) is the specific heat at constant pressure, \( \sigma \) is the fluid electrical conductivity, \( B_0 \) is the magnetic induction, \( k_0 \) is the Visco-elastic parameter, \( k' \) is the permeability of the porous medium, \( Q_0 \) is the dimensional heat absorption coefficient, \( \phi \) is the coefficient of proportionality for the absorption of radiation, \( k \) is the thermal conductivity parameter, \( D \) is the mass diffusivity, \( g \) is the gravitational acceleration, \( \beta_T \) and \( \beta_c \) are the thermal and concentration expansion coefficient, respectively and \( K_1 \) is the chemical reaction coefficient. The magnetic and viscous dissipations are neglected in this study. The third and fourth terms on the RHS of the energy Eq. (3) represent the heat and radiation absorption effects, respectively. It is assumed that \( k' \) is perpendicular to the plate and \( t' \) is the dimensional time.

Under these assumptions, the appropriate boundary conditions for the velocity, temperature and concentration fields are

\[
\begin{align*}
\mathbf{u'} &= u_p', \quad T' = T_w + \varepsilon (T_w - T_\infty) e^{\nu y'}, \quad C' = C_w + \varepsilon (C_w - C_\infty) e^{\nu y'} \quad \text{at} \quad y' = 0 \\
\mathbf{u'} &= 0, \quad T' \to T_\infty, \quad C' \to C_\infty, \quad \text{at} \quad y' \to \infty
\end{align*}
\]

(5)

Where \( u_p' \) is the wall dimensional velocity, \( n' \) is constant. It is clear from Eq. (1) that the suction velocity at the plate surface is a function of time only. Assuming that the following exponential form:

\[
v' = -V_0 (1 + \varepsilon A e^{\nu y'}).
\]

(6)

Where \( A \) is a suction parameter, \( \varepsilon \) and \( A \) are small such that \( \varepsilon << 1 \), \( A << 1 \) and \( V_0 \) is a scale of suction velocity which has non-zero position constant. Now the following non-dimensional variables are introduced.

\[
F = \frac{4y'I'}{\rho C_p V_0^2}, \quad u = \frac{u'}{V_0}, \quad v = \frac{v'}{V_0}, \quad y = \frac{y'}{V_0}, \quad t = \frac{V_0^2 t'}{\nu'},
\]

\[
\begin{align*}
u_p' &= \frac{u_p'}{V_0}, \quad n = \frac{n'V_0^2}{V_0^2}, \quad \theta = \frac{T' - T_\infty}{T_w - T_\infty}, \quad C = C' - C_\infty \quad \frac{C_w - C_\infty}{C_w - C_\infty}
\end{align*}
\]

(7)

In view of the above non-dimensional variables, the basic field Eqn. (2)-(4) can be expressed in the non-dimensional form as

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} - (1 + \varepsilon A e^{\nu y'}) \frac{\partial \mathbf{u}}{\partial y} &= \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 \mathbf{u}}{\partial y'^2} - \left( M + \frac{1}{K} \right) \mathbf{u} + G_T \theta + G_m C - E \left[ \frac{\partial^3 \mathbf{u}}{\partial t^2 \partial y'^2} - (1 + \varepsilon A e^{\nu y'}) \frac{\partial^3 \mathbf{u}}{\partial y'^4} \right] \\
\frac{\partial \theta}{\partial t} - (1 + \varepsilon A e^{\nu y'}) \frac{\partial \theta}{\partial y} &= \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y'^2} - (\phi + F) \theta + Q_c C
\end{align*}
\]

(8)

\[
\begin{align*}
\frac{\partial C}{\partial t} - (1 + \varepsilon e^{\nu y'}) \frac{\partial C}{\partial y} &= \frac{1}{Sc} \frac{\partial^2 C}{\partial y'^2} - \gamma C
\end{align*}
\]

(9)

(10)
\[ u = u_p, \theta = 1 + \varepsilon e^{\alpha t}, C = 1 + \varepsilon e^{\alpha t} \text{ at } y = 0 \] (11)

\[ u \to 0, \theta \to 0, C \to 0 \text{ at } y \to \infty \]

\[ Gr = \frac{\nu g \beta (T_w - T_\infty)}{V_0^3}, \quad Gm = \frac{\nu g \beta (C_w - C_\infty)}{V_0^3} Q_i \frac{\nu Q_i (C_w - C_\infty)}{\rho C_p VT_0 (w - T_\infty)}, \quad Pr = \frac{\nu \rho c_p}{k}, \quad K = \frac{K' V_0^2}{v^2} \] (12)

\[ M \text{ is the magnetic field parameter, } K \text{ is the permeability parameter, } \varepsilon \text{ is the Chemical reaction parameter, } Sc \text{ is the Schmidt number, } \phi \text{ is the heat source parameter and } Q_1 \text{ is the absorption of radiation parameter.} \]

The mathematical statement of the problem is now complete and embodies the solution of Eqs. (8)-(10) Subject to boundary condition (11).

### III. SOLUTION OF THE PROBLEM

Solutions of Eqs. (8)-(10) are obtained by regular and multi parameter perturbation technique. The parameter Visco elastic parameter (E), constant \( \varepsilon \) and \( \lambda \) are assumed small such that \( E \ll 1 \) and \( \varepsilon \ll 1 \)The velocity \( u \), temperature \( C \) within the boundary layer region can be expressed as:

\[ u(y, t) = u_0(y) + \varepsilon e^{\alpha t} u_1(y) + O(\varepsilon^2), \quad \theta(y, t) = \theta_0(y) + \varepsilon e^{\alpha t} \theta_1(y) + O(\varepsilon^2), \quad C(y, t) = C_0(y) + \varepsilon e^{\alpha t} C_1(y) + O(\varepsilon^2) \] (13)

Where \( u_0 \), \( \theta_0 \) and \( C_0 \) are the mean velocity, mean temperature and mean concentration respectively. Using Eq.(13) in Eqs. (8)-(10). Equation harmonic and non-harmonic terms for mean velocity, mean temperature and mean concentration, after neglecting coefficient of \( \varepsilon^2 \), we get Zero order

\[ Eu_0'' + \left(1 + \frac{1}{\beta} \right) u_0'' + u_0' - \left( M + \frac{1}{K} \right) u_0 = -Gr\theta_0 - GmC_0 \] (14)

\[ \theta_0'' + Pr \theta_0' - Pr \phi \theta_0 = -Q_1 Pr C_0 \] (15)

\[ C_0'' + ScC_0' - Sc\gamma C_0 = 0 \] (16)

With corresponding boundary conditions

\[ u_0 = u_p, \theta_0 = 1, C_0 = 1 \text{ at } y = 0 \]

\[ u_0 \to 0, \theta_0 \to 0, C_0 \to 0 \text{ as } y \to \infty \] (17)

First order

\[ Eu_1'' + \left[1 + \frac{1}{\beta} \right] - En \right] u_1'' + u_1' - \left[ M + \frac{1}{K} \right] u_1 = -Gr\theta_1 - GmC_1 - AEu_0'' - Au_0' \] (18)

\[ \theta_1'' + Pr \theta_1' - Pr(\phi + F + n)\theta_1 = -Pr A\theta_0' - Pr Q_1 C_1 \] (19)

\[ C_1'' + ScC_1' - Sc(\gamma + n)C_1 = -AScC_0' \] (20)
With corresponding boundary conditions

\[ u_i = 0, \theta_i = 0, C_i = 0, \text{ on } y = 0 \]
\[ u_i \to 0, \theta_i \to 0, C_i \to 0 \text{ as } y \to \infty \] (21)

Equations (14) and (18) are third order differential equations due to presence of viscoelastic parameter. There are only two boundary conditions. Therefore, it needs one boundary condition more for unique solution. Thus, to avoid this difficulty, we adopted perturbation method and expanded following Beard and Walters [1].

\[ u_0(y) = u_{00}(y) + Eu_{01}(y) + O(E^2) \]
\[ u_1(y) = u_{10}(y) + Eu_{11}(y) + O(E^2) \] (22)

Substitute Eq. (22) in Eq. (14) and equating coefficient of zero order and first order of \( E \), we get

\[ \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u_{00}}{\partial y^2} + \frac{\partial u_{00}}{\partial y} - \left( M + \frac{1}{K} \right) u_{00} = -Gr\theta_0 - GmC_0 \] (23)

\[ \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u_{10}}{\partial y^2} + \frac{\partial u_{10}}{\partial y} - \left( M + \frac{1}{K} \right) u_{10} = -\frac{\partial^2 u_{00}}{\partial y^2} \] (24)

The corresponding boundary conditions are

\[ u_{00} = u_p, \theta_{01} = 0, \text{ on } y = 0 \]
\[ u_{01} \to 0, \theta_{01} \to 0, \text{ as } y \to \infty \] (25)

Substituting Eq. (22) in Eq. (18) and equating coefficient of zero order and first order of \( E \), we get

\[ \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u_{10}}{\partial y^2} + \frac{\partial u_{10}}{\partial y} - \left( M + \frac{1}{K} + n \right) u_{10} = -Gr\theta_1 - GmC_1 - A \frac{\partial u_{00}}{\partial y} \] (26)

\[ \left[ \left( 1 + \frac{1}{\beta} \right) - n \right] \frac{\partial^2 u_{11}}{\partial y^2} + \frac{\partial u_{11}}{\partial y} - \left( M + \frac{1}{K} + n \right) u_{11} = -A \frac{\partial^3 u_{00}}{\partial y^3} - A \frac{\partial u_{10}}{\partial y} - n \frac{\partial^2 u_{10}}{\partial y^2} \] (27)

Where,

\[ B = 1 + \frac{1}{\beta}, \quad S = M + \frac{1}{K} \]

The corresponding boundary conditions are

\[ u_{10} = 0, \theta_{11} = 0, \text{ on } y = 0 \]
\[ u_{10} \to 0, \theta_{11} \to 0, \text{ as } y \to \infty \] (28)

Solving Eqs. (23), (24), (26) and (27) by using the boundary conditions (25) and (28), we get

\[ u_{00} = A_3 e^{-m_y} + A_{12} e^{-m_y} + A_{13} e^{-m_y} \]
\[ u_{01} = A_3 e^{-m_y} + A_{12} e^{-m_y} + A_{13} e^{-m_y} + A_9 e^{-m_y} \]
\[ u_{10} = A_{33} e^{-m_y} + A_{25} e^{-m_y} + A_{10} e^{-m_y} + A_{11} e^{-m_y} + A_{32} e^{-m_y} \]
\[ u_{11} = A_9 e^{-m_9 y} + A_{33} e^{-m_{33} y} + A_{35} e^{-m_{35} y} + A_{35} e^{-m_{35} y} + A_{36} e^{-m_{36} y} + A_{36} e^{-m_{36} y} \]

\[ u_0(y) = u_{00}(y) + Eu_{01}(y) \]

\[ u_0(y) = (A_9 e^{-m_9 y} + A_{12} e^{-m_{12} y} + A_{15} e^{-m_{15} y}) + E(A_{20} e^{-m_{20} y} + A_{17} e^{-m_{17} y} + A_{18} e^{-m_{18} y} + A_9 e^{-m_9 y}) \]

\[ u_i(y) = u_{i0}(y) + Eu_{i1}(y) \]

\[ u(y, t) = u_0(y) + \varepsilon e^{\alpha y} u_1(y) \]  \hspace{1cm} (29)

\[ C(y, t) = C_0(y) + \varepsilon e^{\alpha y} C_1(y) \]

\[ C_0(y) = e^{-m_0 y} \]

\[ C_1(y) = (A_0 e^{-m_{0} y} + A_0 e^{-m_{0} y}) \]

\[ C(y, t) = e^{-m_0 y} + \varepsilon e^{\alpha y} (A_0 e^{-m_{0} y} + A_0 e^{-m_{0} y}) \]  \hspace{1cm} (30)

\[ \theta(y, t) = \theta_0(y) + \varepsilon e^{\alpha y} \theta_1(y) \]

\[ \theta_0(y) = (A_0 e^{-m_{0} y} + A_0 e^{-m_{0} y}) \]

\[ \theta_1(y) = (A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y}) \]

\[ \theta(y, t) = (A_0 e^{-m_{0} y} + A_0 e^{-m_{0} y}) + \varepsilon e^{\alpha y} (A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y} + A_1 e^{-m_{1} y}) \]  \hspace{1cm} (31)

### 3.1 Skin Friction

Very important physical parameter at the boundary is the skin friction which is given in the non-dimensional form and derived as

\[ \tau = \frac{\tau_w}{\sqrt{\rho U_0}} \]

\[ = -(m_9 A_{36} + m_3 A_{12} + m_1 A_{45}) - E(m_6 A_{20} + m_3 A_{17} + m_3 A_{38} + m_4 A_{19}) \]

\[ - \varepsilon e^{\alpha y} \left[(m_1 A_{31} + m_1 A_{37} + m_1 A_{12}) + m_4 A_{10} + m_4 A_{11} + m_4 A_{32}ight] + E(m_6 A_{59} + m_7 A_{33} + m_6 A_{31} + m_6 A_{54} + m_4 A_{55} + m_3 A_{56} + m_3 A_{57} + m_3 A_{58}) \]  \hspace{1cm} (32)

### 3.2 Rate of Heat Transfer

Physical parameter like rate of heat and mass transfer in the form of Nusselt number derived is given below.

\[ Nu = X \left. \frac{\hat{\partial T}}{\hat{\partial y}} \right|_{y=0} = Nu Re_x^{-1} \left. \frac{-\hat{\partial \theta}}{\hat{\partial y}} \right|_{y=0} \]

\[ = Re_x (m_1 A_1 + m_1 A_2) + \varepsilon e^{\alpha y} (m_1 A_1 + m_1 A_3 + m_2 A_7 + m_1 A_7) \]

\hspace{1cm} (33)
Where \( \text{Re}_X \) is the Reynolds number.

### 3.3 Sherwood Number

Another physical parameter like rate of mass transfer in the form of Sherwood number derived is given below.

\[
\frac{\partial C}{\partial y} \bigg|_{y=0} = \frac{\partial C}{\partial y} \bigg|_{y=0} = \text{Sh}^{-1} \frac{\partial C}{\partial y} = \frac{\partial C}{\partial y} \bigg|_{y=0} = \text{Sh} \left[ \frac{m_1 + \varepsilon \exp(nt)(m_1A_1 + m_2A_2)}{m_1 + \varepsilon \exp(nt)(m_1A_1 + m_2A_2)} \right]
\]

### IV. RESULT AND DISCUSSION

In this section, the effects of various physical parameters such as Visco-elastic parameter (E), Grashof number (Gr), modified Grashof number (Gm), Magnetic parameter (M), Prandtl number (Pr), Heat absorption coefficient (\( Q_1 \)), Heat source parameter (\( \varphi \)), Schmidt number (Sc), Chemical reaction coefficient (\( \gamma \)), Permeability parameter (K) are analyzed. The analytical results obtained in the previous section are studied numerically and the variations in velocity \( u(y, t) \), temperature \( \theta(y, t) \), concentration \( \varphi(y, t) \) are discussed through graphs. Also the variation of Skin friction, Nusselt number and Sherwood number are discussed for various values of M, K, E, A, \( \gamma \), Gr, Gm, Sc, \( Q_1 \), \( \varphi \).

**Fig. 1** describes the variations in velocity distribution with respect to the magnetic parameter M. This figure describes that the velocity decreases as usual with an increase in M.

**Fig. 2** and **Fig. 10** Describes the effect of Prandtl number Pr on the velocity and temperature distributions. This figure shows that the velocity and temperature decrease with an increase in Pr.

**Fig. 3** Display the effect of Grashof number Gr on velocity distribution. This figure shows that velocity and temperature decrease with an increase in Gr.

**Fig. 4** depicts the effects of Magnetic parameter F on velocity distribution. It describes that velocity increases with an increase in F.

**Fig. 5** represents the effects of modified Grashof number Gm on velocity distribution. This figure shows that velocity increases with an increase Gm.

**Fig. 6** describes the effects of Casson parameter \( \beta \) on velocity distribution. The figure shows that velocity increase with a decrease in \( \beta \).

**Fig. 7** Display the effects of Heat source parameter \( \varphi \) on the velocity distribution. This figure describe that velocity increases with a decrease in \( \varphi \).

**Fig. 8** and **Fig. 12** Display the effect of Sc on temperature and concentration distributions. It shows that temperature and concentration increase with a decrease in Sc.

**Fig. 9** and **Fig. 10** Illustrates the effects of \( \gamma \) on the temperature and concentration distributions. It describes that the temperature and concentration increase with a decrease in \( \gamma \).

**Fig. 1** Effects of M on velocity profiles.
Fig. 2 Effects of Pr on velocity profiles.

Fig. 3 Effects of Gr on velocity profiles.
**Fig. 4** Effects of F on velocity profiles.

**Fig. 5** Effects of Gm on velocity profiles.
Fig. 6 Effects of $\beta$ on velocity profiles.

Fig. 7 Effects of $\phi$ on velocity profiles.
Fig. 8 Effects of Sc on temperature profiles.

Fig. 9 Effects of γ on temperature profiles.
Fig. 10 Effects of Pr on temperature profiles.

Fig. 11 Effects of $\gamma$ on concentration profiles.
V. CONCLUSIONS

➢ Velocity increases for increasing values of Grashof number (Gr), Magnetic parameter (F), solutal Grashof number (Gm).

➢ Velocity increases for decreasing values of magnetic field parameter (M), Prandtl number (Pr), Casson parameter ($\beta$), Heat source parameter ($\emptyset$).

➢ Temperature increases for increasing values of Schmidt number (Sc), Chemical reaction parameter ($\gamma$), Prandtl number (Pr).

➢ Concentration increases for decreases the value of Chemical reaction parameter ($\gamma$), Schmidt number (Sc).

NOMENCLATURE

$x'$ Dimensional distances

$y'$ Perpendicular to the plate

$t'$ Dimensional time

$u'$ Components of dimensional velocity $x'$

$v'$ Components of dimensional velocity $y'$

$T'$ Dimensional temperature

$C'$ Dimensional concentration

$C_w$ Concentration at the wall

$T_w$ Temperature at the wall

$C_\infty$ Free stream dimensional concentration

$T_\infty$ Free stream dimensional temperature

$\rho$ Fluid density

$\nu$ Kinematic viscosity
Specific heat at constant pressure
Fluid electrical conductivity
Magnetic induction
Visco-elastic parameter
Permeability of the porous medium
Dimensional heat absorption coefficient
Coefficient of proportionality for the absorption of radiation
Thermal conductivity parameter
Mass diffusivity
Chemical reaction parameter
Gravitational acceleration
Casson parameter
Heat source of parameter
Thermal expansion coefficient
Concentration expansion coefficient
Chemical reaction coefficient
Magnetic parameter
Magnetic field parameter

APPENDIX

\[ m_1 = \frac{1}{2} \left( Sc + \sqrt{Sc^2 + 4Sc\gamma} \right), \quad m_2 = \frac{1}{2} \left( Sc + \sqrt{Sc^2 + 4Sc(\gamma + n)} \right) \]
\[ m_3 = \frac{1}{2} \left( \Pr + \sqrt{\Pr^2 + 4Pr(\phi + F)} \right), \quad m_4 = \frac{1}{2} \left( \Pr + \sqrt{\Pr^2 + 4Pr(\phi + F + n)} \right) \]
\[ m_5 = \frac{1}{2B} \left( 1 + \sqrt{1 + 4BS} \right), \quad m_6 = \frac{1}{2B} \left( 1 + \sqrt{1 + 4BS} \right) \]
\[ m_7 = \frac{1}{2B} \left( 1 + \sqrt{4(B - n)(M + \frac{1}{K} + n)} \right), \quad m_8 = \frac{1}{2(B - n)} \left( 1 + \sqrt{4(B - n)(M + \frac{1}{K} + n)} \right) \]

\[ A_1 = \frac{ASCm_1}{m_2^2 - Scm_1 - Sc(\gamma + n)}, \quad A_2 = 1 - A_1, \quad A_3 = \frac{APEA_3m_3}{m_4^2 - Prm_3 - Pr(\phi + F + n)}, \quad A_4 = 1 - A_3 \]
\[ A_5 = \frac{APEA_5m_5}{m_6^2 - Prm_5 - Pr(\phi + F + n)}, \quad A_6 = \frac{APEA_6m_6}{m_7^2 - Prm_6 - Pr(\phi + F + n)} \]
\[ A_7 = \frac{-Q_1 Pr A_7}{m_8^2 - Prm_8 - Pr(\phi + F + n)}, \quad A_8 = \frac{-Q_1 Pr A_8}{m_9^2 - Prm_9 - Pr(\phi + F + n)} \]
\[ A_9 = A_6 + A_8, \quad A_{10} = A_4 + A_7 + A_9, \quad A_{11} = 1 - A_{10} \]
\[ A_{12} = \frac{-GrA_{12}}{Bm_2^2 - m_2 - \left( M + \frac{1}{K} \right)}, \quad A_{13} = \frac{-GrA_{13}}{Bm_3^2 - m_3 - \left( M + \frac{1}{K} \right)} \]
\[ A_{14} = \frac{-Gm}{Bm_4^2 - m_4 - \left( M + \frac{1}{K} \right)}, \quad A_{15} = A_{13} + A_{14}, \quad A_{16} = u_p - (A_{12} + A_{15}) \]
\[ A_{17} = \frac{-A_{16}(m_3^3)}{Bm_5^2 - m_5 - \left( M + \frac{1}{K} \right)}, \quad A_{18} = \frac{-A_{12}(m_1^3)}{Bm_6^2 - m_6 - \left( M + \frac{1}{K} \right)} \]
\[
\begin{align*}
A_{19} &= \frac{A_{15}(m_5)^3}{Bm_4^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{21} &= -GrA_{1}, \\
A_{23} &= -GrA_{3}, \\
A_{25} &= -GmA_{2}, \\
A_{27} &= AA_{16}m_3, \\
A_{29} &= AA_{15}m_4, \\
A_{31} &= A_{23} + A_{25}, \\
A_{32} &= A_{24} + A_{26} + A_{29}, \\
A_{33} &= -(A_{27} + A_{21} + A_{30} + A_{31} + A_{32}) \\
A_{34} &= AA_{16}(m_5)^3, \\
A_{35} &= AA_{15}(m_4)^3, \\
A_{36} &= \frac{AA_{16}(m_5)^3}{m_2^2 - m_5 - \left(M + \frac{1}{K}\right)}, \\
A_{37} &= \frac{AA_{15}(m_4)^3}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{38} &= \frac{AA_{16}(m_5)^3}{m_2^2 - m_5 - \left(M + \frac{1}{K}\right)}, \\
A_{39} &= \frac{AA_{15}(m_4)^3}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{41} &= \frac{AA_{15}(m_4)^3}{m_2^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{42} &= \frac{AA_{16}(m_5)^3}{m_2^2 - m_5 - \left(M + \frac{1}{K}\right)}, \\
A_{43} &= \frac{AA_{15}(m_4)^3}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{44} &= \frac{AA_{16}(m_5)^3}{m_2^2 - m_5 - \left(M + \frac{1}{K}\right)}, \\
A_{45} &= \frac{AA_{15}(m_4)^3}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{46} &= \frac{nA_{33}(m_5)^2}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{48} &= \frac{nA_{27}(m_5)^2}{m_1^2 - m_4 - \left(M + \frac{1}{K}\right)}, \\
A_{20} &= -(A_{17} + A_{18} + A_{19}).
\end{align*}
\]
\[
A_{50} = \frac{nA_{30}(m_3)^2}{m_3^2 - m_3 - \left(\frac{1}{M + \frac{1}{K}} + n\right)}, \quad A_{51} = \frac{nA_{31}(m_1)^2}{m_2^2 - m_2 - \left(\frac{1}{M + \frac{1}{K}} + n\right)}
\]
\[
A_{52} = \frac{nA_{32}(m_1)^2}{m_1^2 - m_1 - \left(\frac{1}{M + \frac{1}{K}} + n\right)}, \quad A_{53} = A_{41} + A_{47}
\]
\[
A_{54} = A_{44} + A_{45} + A_{46}, \quad A_{55} = A_{47} + A_{48}, \quad A_{56} = A_{49} + A_{50} + A_{51} + A_{52}
\]
\[
A_{57} = A_{53}, \quad A_{58} = A_{54} + A_{55} + A_{56} + A_{57} + A_{58}
\]

VI. REFERENCES


