Basic Techniques and Results on q-k-hermitian doubly Stochastic matrices and q-s- hermitian doubly Stochastic matrices

Dr.Gunasekaran K. and Mrs.Seethadevi R.
Department of Mathematics, Government arts College (Autonomous), Kumbakonam, Tamilnadu, India.

INTRODUCTION:

Ann Lee [1] has initiated the study of secondary hermitian matrices, and q-s-k hermitian doubly stochastic matrices are developed in quaternion matrices. Ann Lee [1] has initiated the study of secondary hermitian matrices, that is matrices whose entries are hermitian about the secondary diagonal. Ann Lee [1] has show that the matrix A, the usual transpose A^T and Secondary transpose $A^θ$ are related as $A^θ=VA^TV$ and $A^T=VA^θV$. where V is a modified unit matrix with units in the secondary diagonal.

Abstract:

The basic concepts and theorems of q-k-hermitian doubly stochastic matrices and q-s- hermitian doubly stochastic matrices are introduced with examples.

Keywords:

q-k-hermitian doubly stochastic matrix, q-s- hermitian doubly stochastic matrix.

I. INTRODUCTION:

Ann Lee [1] has initiated the study of secondary hermitian matrices, and q-k hermitian and q-s- hermitian doubly stochastic matrices are developed in quaternion matrices. Ann Lee [1] has initiated the study of secondary hermitian matrices, that is matrices whose entries are hermitian about the secondary diagonal. Ann Lee [1] has show that the matrix A, the usual transpose A^T and Secondary transpose $A^θ$ are related as $A^θ=VA^TV$ and $A^T=VA^θV$. where V is a modified unit matrix with units in the secondary diagonal.

Definition:

A matrix $A \in H^{n\times n}$ is said to be q-k-hermitian doubly stochastic matrix if A=KA*K.

Theorem: 1

If $A \in H^{n\times n}$ is q-k-hermitian doubly stochastic matrix then A=KA*K.

Proof:

KA*K=KAK where A*=A.
=AKK where KA=AK.
$=AK^2$ where $K^2=I$.

Theorem:2

Let $A^* \in H^{n\times n}$ is q-k-hermitian doubly stochastic matrix then A*=KAK.

Proof:

A*=(KA*K)=KAK = KA*K where A=A*
=A*KK where KA*=A*K
$=A^*K^2$ where $K^2=I$

Theorem:3

Let $A, B \in H^{n\times n}$ q-k-hermitian doubly stochastic matrix then ½ (A+B) is q-k-hermitian doubly stochastic matrix.

Proof:

Let A and B are q-k-hermitian doubly stochastic matrix if A=KA*K and B=KB*K. To prove ½ (A+B) is q-k-hermitian doubly stochastic matrix we will show that

$$\frac{1}{2}(A+B)=K\frac{1}{2}(A+B)^*K.$$

Now $K\frac{1}{2}(A+B)^*K = K\frac{1}{2}(A^*+B^*)K = \frac{1}{2}K(A^*+B^*)K$

$$=\frac{1}{2}(KA^*+KB^*)K$$

Where KA*K=A $\quad =\frac{1}{2}(KA^*K+KB^*K)$

KB*K=B $\quad =\frac{1}{2}(A+B).$ 2

Theorem: 4 [5]

Any q-k-hermitian doubly stochastic matrix can be represent as sum of q-k-hermitian doubly stochastic matrix and skew q-k-hermitian doubly stochastic matrix.

Proof:

To prove that $\frac{1}{2}(A+KA^*K)$ and $\frac{1}{2}(A-KA^*K)$ are q-k-hermitian doubly stochastic 2 matrices. We will show that. $\frac{1}{2}(A+KA^*K)=K\frac{1}{2}(A+KA^*K)^*K.$ and

$$\frac{1}{2}(A-KA^*K)=K\frac{1}{2}(A-KA^*K)^*K.$$

Using (theorem 3)

$K\frac{1}{2}(A+KA^*K)^*K=\frac{1}{2}(A+KA^*K)^*K$ and $K\frac{1}{2}(A-KA^*K)^*K=\frac{1}{2}(A-KA^*K)^*K$

Then $\frac{1}{2}(A+KA^*K)+\frac{1}{2}(A-KA^*K)=\frac{2A}{2}=A$ Hence proved.

Theorem: 5

If A and B are q-k-hermitian doubly stochastic matrix then AB is not an q-k-hermitian doubly stochastic matrix.

Proof:

Let A and B are q-k-hermitian doubly stochastic matrix if A=KA*K and B=KB*K. Since A* and B* are also q-k-hermitian doubly stochastic matrices then

A*=KAK and B*=KBK.

To prove AB is not an q-k-hermitian doubly stochastic matrix. We will show that AB=K(AB)*K

K(AB)*K=KB*A*K=K(KBK)(KAK)K

Where A*=KAK and B*=KBK

$= K^2BK^2AK$. Where $K^2=I$

$=BA \neq AB$ where $AB \neq BA$.

Since Quaternion matrices does not satisfy commutative property.

Remark 5(a) Since theorem (5) the product of doubly stochastic matrices is doubly stochastic but not an k-hermitian and also q-k-hermitian doubly stochastic matrix.

Theorem: 6

If A and B are q-k-hermitian doubly stochastic matrices and K is the modified unit matrix, K={(1) (2 3)}. Then KA is also q-k-hermitian doubly stochastic matrix.

Proof:

Let A and B are q-k-hermitian doubly stochastic matrix if A=KA*K and B=KB*K. Since A*and B* are also q-k-hermitian doubly stochastic matrices then A*=KAK and B*=KBK.

To prove KB is q-k-hermitian doubly stochastic matrix. We will show that KA=K(KA)*K

Now K(KA)*K=K(A*K*)K=KA*K*K=

=KA where KA* where KA*=KA. K*K=I

RESULT:

For $A \in H^{n \times n}$ is q-k-hermitian doubly stochastic matrices for the following are equivalent.

[1] A=KA*K.

[2] KA is q-k-hermitian doubly stochastic matrix.

[3] AK is q-k-hermitian doubly stochastic matrix. $(1) \Rightarrow (2)$

KA*K=A*KK where KA*=A*K

$=A^*K^2$

=A*=A where KA=AK=A.

=KA.

$(2) \Rightarrow (3)$

KA=KA*

=A*K where KA*=A*K

=AK.

$(3) \Rightarrow (1)$

AK=A*K

= AK=A. where KA=AK=A.

(where k is a modified unit matrix units in the Secondary diagonal)

Example:

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}$$

$$\Rightarrow \quad k = (1)(2\ \ 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(i)\ \text{KA}^* K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2-i+j & -2+1-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$= A$$

$similarly\ KAK = A^*$

$$\text{(ii) KA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2-i+j & 2+i-j \\ 2-i+j & -4+i-j & 3 \\ -2+i-j & 7 & -4-i+j \end{pmatrix}$$

$$= (KA)^*$$

KA is q-s hermitian doubly stochastic matrix.

Similarly KA* is also q-s hermitian doubly stochastic matrix.

$$\text{(iii) AK} = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2-i+j & 2+i-j \\ 2-i+j & -4+i-j & 3 \\ -2+i-j & 7 & -4-i+j \end{pmatrix}$$

$$= (AK)^*$$

AK is q-s hermitian doubly stochastic matrix.

Similarly A^*K is also q-s hermitian doubly stochastic matrix.

Definition:

A matrix $A \in H^{n\times n}$ is said to q-s-hermitian doubly stochastic matrix if $A^\theta = VA^*V$ where V is a modified unit matrix with units in the secondary diagonal.

Theorem: 7

Let $A \in H^{n\times n}$ is q-s-hermitian doubly stochastic matrix then $A^\theta = VA^*V$.

Proof: $\left[\because A^* = VA^\theta V\right]$

$$VA^*V = V\left(VA^\theta V\right)V = V^2A^\theta V^2$$

$= A^\theta$ where $V^2 = I$.

Theorem: 8

Let $A^* \in H^{n\times n}$ is q-s-hermitian doubly stochastic matrix then $A^* = VA^\theta V$

Proof:

$= VA^\theta V = V(VA^*V)V = V^2A^*V^2$

$= A^*$ where $V^2 = I$

Theorem: 9

Let $A, B \in H^{n\times n}$ is q-s-hermitian doubly stochastic matrix then $\frac{1}{2}(A+B)$ is q-s-hermitian doubly stochastic matrix. Proof:

Let A and B are q-s-hermitian doubly stochastic matrices if $A^\theta = VA^*V$ and $B^\theta = VB^*V$

To prove $\frac{1}{2}(A+B)$ is q-s-hermitian doubly stochastic matrix we will showthat

$$\frac{1}{2}(A+B)^\theta = V\frac{1}{2}(A+B)^*V$$

Now $V\frac{1}{2}(A+B)^*V = V\frac{1}{2}\left(A^* + B^*\right)V$

$$= \frac{1}{2}V\left(A^* + B^*\right)V$$

$$=\frac{1}{2}\left(VA^{*}+VB^{*}\right)V$$
$$=\frac{1}{2}\left(VA^{*}V+VB^{*}V\right)$$
$$=\frac{1}{2}\left(A^{\theta}+B^{\theta}\right)$$

Where,
$$A^{\theta}=VA^{*}V$$
$$B^{\theta}=VB^{*}V$$

$$=\frac{1}{2}\left(A+B\right)^{\theta}$$

Theorem: 10

If A and B are q-s-hermitian doubly stochastic matrices then AB is not an q-s- hermitian doubly stochastic matrix.

Proof:

Let A and B are q-s-hermitian doubly stochastic matrices if $A^{\theta}=VA^{*}V$ and $B^{\theta}=VB^{*}V$ Since A^{*} and B^{*} are also q-s-hermitian doubly stochastic matrices then $A^{*}=VA^{\theta}V$ and $B^{*}=VB^{\theta}V$

To prove AB is q-s-hermitian doubly stochastic matrix we will show that

$$\left(AB\right)^{\theta}=V\left(AB\right)^{*}V=V\left(VB^{\theta}V\right)\left(VA^{\theta}V\right)V$$

Where, $A^{*}=VA^{\theta}V$ and $B^{*}=VB^{\theta}V$

$$=V^{2}B^{\theta}V^{2}A^{\theta}V^{2}=B^{\theta}A^{\theta}$$

Where, $\neq\left(AB\right)^{\theta}$ Where $AB\neq BA.\ V^{2}=I$

Quaternions does not satisfy commutative property

Theorem:11

If A is q-s-hermitian doubly stochastic matrix and V is a modified unit matrix with units in the secondary diagonal then VA is also. q-s hermitian doubly stochastic matrix.

Proof:

Let A is q-s-hermitian doubly stochastic matrices if $A^{\theta}=VA^{*}V$ Since A* is q-s hermitian doubly stochastic matrices then $A^{*}=VA^{\theta}V$

To prove VA is q-s-hermitian doubly stochastic matrix we will show that.

$$(VA) = V(VA)\,V^{*}$$

Now $V(VA)^{*}V = V(A^{*}V^{*})V = V(VA^{\theta}V)V^{2}$

$$= A^{\theta}V^{\theta} \text{ where}$$

$$= (VA)^{\theta}V^{2} = I$$

Result:

For $A \in H^{n\times n}$ is q-s-hermitian doubly stochastic matrix for the following are equivalent.

[1] $A^{\theta} = VA^{*}V$

[2] VA is q-s-hermitian doubly stochastic matrix.

[3] AV is q-s -hermitian doubly stochastic matrix.

(1) $\Rightarrow$ (2) $A^{\theta} = VA^{*}V$

$$= V(VA^{\theta}V)VA^{*} = VA^{\theta}V$$

$$= V^{2}A^{\theta}V^{2} \text{ Where } V^{2} = I$$

$$= A \quad A^{\theta} = A = VA^{*}V$$

$$= VA \text{ Where } VA = AV = A$$

$(2) \Rightarrow (3)\; VA = V(VA^{\theta}V)$

$$= V^{2}A^{\theta}V$$

Where $V^{2} = I$

$$= A^{\theta}V$$

Where $A^{\theta} = A = A^{*} = VA^{\theta}V$

$$= AV$$

Example:

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$A^{*} = A^{\theta} = \begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{(ii)}VA^{*}V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}$$

$$= A^{\theta}$$

$$\text{(ii)}VA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -2-i+j & 2+i-j & 1 \\ -4+i-j & 3 & 2-i+j \\ 7 & -4-i+j & -2+i-j \end{pmatrix}$$

$$= (VA)^{*}$$

$\Rightarrow$ VA is q-s-hermitian doubly stochastic matrix.

Similarly $A^{*}V$ is also q-s-hermitian doubly stochastic matrix.

$$\text{(iii) AV} = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2-i+j & 2+i-j & 1 \\ -4+i-j & 3 & 2-i+j \\ 7 & -4-i+j & -2+i-j \end{pmatrix} = (AV)$$

$\Rightarrow$ AV is q-s hermitian doubly stochastic matrix.

Similarly $A^{*}V$ is also q-s hermitian doubly stochastic matrix.

REFERENCES:

[1] Ann.Lee. Secondary symmetric and Skew Symmetric Secondary, orthogonal matrices, period, math Hungary, 7, 63-70. (1976).

[2] Hill, R.D, and waters, S.R: On K-real and k-hermitian matrices; Lin. Alg. Appl., 169, 17-29 (1992).

[3] G.latoucke, V.Ramaswami, Introduction to matrix Analytic methods in Stochastic modeling, 1st edition, Chapter2: PH distributions; ASASIAM, 1999.

[4] S.Krishnamoorthy, K.Gunasekaran, N.Mohana, "Characterization and theorems on doubly stochastic matrices".
Antartica Journal of mathematics, 11 (5) (2014).

[5] K.Gunasekaran, N.Mohana and R.Seethadevi, "Characterization and theorems

on quaternion doubly stochastic matrices" journal of ultra scientist and physical science" (2017).