Discrete dynamics of prey-predator system

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Abstract: *This research manuscript deals with investigation of the dynamics of discrete-time prey-predator model. The Stability analysis of the discrete model is discussed at all the fixed points. Furthermore, particular conditions dealing with the existence of flip bifurcation and hopf bifurcation are discussed.*

Keywords: Flip bifurcation, forward Euler method, fixed points, Prey-predator system.

1.Introduction

Great number of ecologists and mathematicians showing keen interest in the area of prey-predator modeling, in present times. Dynamics of the prey-predator system in ecology, is studied and discussed by many ecologists. Thorough their studies they contributed significantly for the development and growth of continuous models for large size populations [1-13]. As far as small size population is considered, presently available discrete-time models are sufficiently appropriate and make available efficient results [18-26]. Keeping in mind available literature, this piece of research work will study the stability of discrete-time prey-predator system.

Consider a prey-predator model of the form

$$\begin{cases} \frac{dx}{dt} = ax(1-x) - bxy, \\ \frac{dy}{dt} = mbxy - dy, \end{cases} \quad (1)$$

here the densities of prey and predator populations are given by $x\ (t)$ and $y\ (t)$ respectively, the intrinsic growth rate of prey and predator are denoted by a, b respectively. Also, d denotes natural death of predator species and m denotes the conversion rate for predator in a particular habitat.

On applying forward Euler's scheme to the system of equations in (1), the discrete-time system is obtained as follows:

$$\begin{cases} x \to x + \delta[ax(1-x) - bxy], \\ y \to y + \delta(mbxy - dy), \end{cases} \quad (2)$$

where δ is the step size.

Stability of the fixed points

The fixed points of the system (2) are O(0,0), A(1,0) and C(x^* , y^*) , where x^*, y^* satisfy

$$\begin{cases} a(1-x^*) - by^* = 0, \\ \quad mbx^* - d = 0, \end{cases} \qquad (3)$$

The Jacobian matrix of (2) at the fixed point (x, y) is written as

$$J=\begin{bmatrix} 1+\ \delta\,(a-2ax-by) & -\delta bx \\ \delta mby & 1+\ \delta\,(mbx-d) \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is given by

$$\lambda^2 + \mathrm{p}(x,y)\,\lambda + \mathrm{q}\,(x,y) = 0, \qquad (4)$$

where

$$\mathrm{p}(x,y) = -\,\mathrm{tr}(J) = -2 - \delta\,(\,(a-2ax-by+mdx-d),$$

$$\mathrm{q}(x,y) = \det J = [\,1+\delta\,(a-2ax-by)][\,1+\delta\,(mbx-d)] + \delta^2 \mathrm{m}b^2\mathrm{xy}.$$

Now we state a lemma as similar as in [14-17]:

Lemma 2.1. Let $F(\lambda) = \lambda^2 + B\,\lambda + C$. Suppose $F(1) > 0$; λ_1 and λ_2 are roots of $F(\lambda) = 0$. Then, we have

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff $F(-1) > 0$ and $C < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) iff $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ iff $F(-1)>0$ and $C> 1$;

(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ iff F(-1) and $B \neq 0,2$;

(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ iff $B^2 - 4C < 0$ and $C = 1$.

Let λ_1 and λ_2 be the roots of (4), which are the eigenvalues of the fixed point (x, y). The fixed point (x,y) is a sink or locally asymptotically stable if, $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The fixed point (x,y) is a source or locally unstable if, $|\lambda_1| > 1$ and $|\lambda_2| > 1$. The fixed point (x,y) is non-hyperbolic if, either $|\lambda_1| = 1$ or $|\lambda_2| = 1$. The fixed point (x,y) is a saddle if, $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

Proposition 2.2. The fixed point O (0,0) is a sink.

The Jacobian matrix of (2) at O (0,0) is given by

$$J = \begin{bmatrix} 1+\ \delta a & 0 \\ 0 & 1-\ \delta d \end{bmatrix}.$$

The eigen values are $1 + \delta a$, $1 - \delta d$. Now

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ $then\ \frac{-2}{a} < \delta < 0\ and\ 0 < \delta < \frac{2}{d}$.

Therefore O(0,0) is a sink for $\frac{-2}{a} < \delta < \frac{2}{d}$.

(ii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *then* $\delta > 0$ *or* $\delta < \frac{-2}{a}$ and $\delta < 0$ *or* $\delta > \frac{2}{d}$.

Therefore O(0,0) is not a source .

(iii) $|\lambda_1| = 1$ or $|\lambda_2| = 1$ then $\delta = 0$ or $\delta = \frac{-2}{a}$ or $\delta = 0$ or $\delta = \frac{-2}{d}$.

Therefore O(0,0) is not non-hyperbolic .

(iv) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ *or* $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Then $\delta > 0$ *or* $\delta < \frac{-2}{a}$ and $o < \delta < \frac{2}{d}$ or $\frac{-2}{a} < \delta < 0$ and $\delta < 0$ or $\delta > \frac{2}{d}$. Which is not possible. Therefore O (0, 0) is not saddle.

Proposition 2.3. The fixed point A(1,0) is not a sink, a source if, $\delta > \frac{2}{a}$, non-hyperbolic if, $\delta = 0$, a saddle if, $0 < \delta < \frac{2}{a}$.

At (1,0) the Jacobian matrix of (2) is

$$J = \begin{bmatrix} 1-\delta a & -\delta b \\ 0 & 1+\delta(-d+mb) \end{bmatrix}$$

The eigen values are $1 - \delta a$, $1 + \delta(-d+mb)$. Here

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ then $\frac{2}{a} < \delta < 0$ and $\frac{-2}{-d+mb} < \delta < 0$, which is not possible Therefore A (1,0) is not a sink.

(ii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then $\delta < 0$ *or* $\delta > \frac{2}{a}$ and $\delta > 0$ *or* $\delta < \frac{-2}{-d+mb}$. Thus A (1, 0) is source if, $\delta > \frac{2}{a}$.

(iii) $|\lambda_1| = 1$ or $|\lambda_2| = 1$ then $\delta = 0$ or $\delta = \frac{2}{a}$ or $\delta = 0$ or $\delta = \frac{-2}{-d+mb}$. Therefore A (1, 0) is non-hyperbolic if, $\delta = 0$.

(iv) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Then $\delta < 0$ or $\delta > \frac{2}{a}$ and

$\frac{-2}{-d+mb} < \delta < 0$ or $\frac{2}{a} > \delta > 0$ and $\delta > 0$ or $\delta < \frac{-2}{-d+mb}$. Therefore A (1,0) is a

saddle if, $0 < \delta < \frac{2}{a}$.

The Jacobian matrix of (2) at the fixed point C(x^* , y^*) is given by

$$J = \begin{bmatrix} 1+\delta(a-2ax^*-by^*) & -\delta bx^* \\ \delta mby^* & 1+\delta(mbx^*-d) \end{bmatrix}.$$

The characteristic equation for the above Jacobian matrix is

$$\lambda^2 + p(x^*,y^*)\,\lambda + q\,(x^*,y^*) = 0, \qquad (5)$$

where $p(x^*,y^*) = -2-\delta(a-2ax^*-by^*+mbx^*-d)$

$= -2-\delta G,$

$q\,(x^*,y^*) = [\,1+\delta(a-2ax^*-by^*)\,]\,[1+\delta(mbx^*-d)\,] + \delta^2 mb^2x^*y^*$

$= 1+\delta G+\delta^2 H,$

and

$G = a - 2ax^* - by^* + mbx^* - d,$

$H = (\,a-2ax^*-by^*)\,(mbx^*-d) + mb^2x^*y^*.$

Now $F(\lambda) = \lambda^2-(2+G\delta)\lambda+(1+G\delta+H\delta^2)$. So $F(1) = H\delta^2$ and $F(-1) = 4+2G\delta+H\delta^2$.

The following proposition can be obtained by using the lemma 2.1,

Proposition 2.4. There exist different topological types of $D(x^*,y^*)$ for all possible parameters.

(i) $D(x^*,y^*)$ is a sink if either condition (i.1) or (i.2) holds:

(i.1) $G^2-4H \geq 0$ and $0 < \delta < \frac{-G-\sqrt{G^2-4H}}{H}$

(i.2) $G^2-4H < 0$ and $0 < \delta < \frac{-G}{H}$.

(ii) $D(x^*,y^*)$ is a source if either condition (ii.1) or (ii.2) holds:

(ii.1) $G^2-4H \geq 0$ and $\delta > \frac{-G+\sqrt{G^2-4H}}{H}$,

(ii.2) $G^2-4H < 0$ and $\delta > \frac{-G}{H}$.

(iii) $D(x^*,y^*)$ is a non-hyperbolic if either condition (iii.1) or (iii.2) h

(iii.1) $G^2-4H \geq 0$ and $\delta = \frac{-G\pm\sqrt{G^2-4H}}{H}$,

(iii.2) $G^2-4H < 0$ and $\delta = \frac{-G}{H}$.

(iv) $D\,(x^*,y^*)$ is saddle for all values of the parameters, except for that values which lie in (i) to (iii).

If the condition (iii.1) of the proposition 2.5 holds, then one of the eigen values of the fixed point D(x^*,y^*) is -1 and the magnitude of the other is not unity. The condition (iii.1) of the proposition 2.5 may be expressed as follows:

F_{D1} = { (a,b,d,m,δ) : $\delta = \frac{-G-\sqrt{G^2-4H}}{H}$, $G^2 - 4H \geq 0$ and a,b,d,m,$\delta > 0$ },

F_{D2} = { (a,b,d,m,δ) : $\delta = \frac{-G-\sqrt{G^2-4H}}{H}$, $G^2 - 4H \geq 0$ and a,b,d,m,$\delta > 0$ }.

If the term (iii.2) of proposition 2.5 holds, then the eigen values of the fixed point D(x^*, y^*) occur as a conjugate pair of complex number with modulus unity. The condition (iii.2) of the proposition 2.5 may be expressed as follows:

H_D = {(a, b, d, m,δ): δ=$\frac{-G}{H}$, G^2-4H<0 and a,b,d,m, $\delta > 0$ }.

Here F_{D1} and $F_{D1}$are the regions of existence of flip bifurcation, and H_D is region for existence of Hopf bifurcation.

Conclusion

Present research study deals with the discrete dynamics of predator-prey system. The stability of the model at all the fixed points has been examined. The maps may undergo flip bifurcation and Hopf bifurcation at the fixed points under specific conditions when δ varies in small neighbourhood of their domain respectively.

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