APPLICATION OF ELZAKI TRANSFORM AND ADOMIAN POLYNOMIAL FOR SOLVING THIRD ORDER AND FIFTH ORDER KDV EQUATIONS WITH POTENTIAL TERM

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Abstract:

The combination of Elzaki transform and Adomian was used to obtain the approximate analytical solutions of KdV equations in this research work. In total, three third order KdV equations and two fifth order KdV equations were considered. Elzaki transform method was applied to obtain the approximate analytical solutions of all the aforementioned equations. Adomian polynomial was introduce as essential tool to linearize all the associated nonlinear terms in the equations since Elzaki transform cannot handle nonlinear terms. All the problem considered yield exact solutions with few iterations.

Keywords: *Nonlinear Differential equations, Elzaki transform, Adomian polynomial, Third order KdV equations, Fifth order KdV equations, Finite difference method*

Introduction

The differential equations have played a central role in every aspect of applied mathematics for every long time and with the advent of the computer, their importance has increased father. Thus investigation and analysis of differential equations cruising in applications led to many deep mathematical problems; therefore, there are so many different techniques in order to solve differential equations.

In order to solve the differential equations, the integral transforms were extensively used and thus there are several words on the theory and applications of integral transforms such as the Laplace, Fourier, Mellin, Hankel and Sumudu, to name but a few. Recently, Tarig Elzaki introduced a new integral transform, named the Elzaki transform, and further applied it to the solution of ordinary and partial differential equations.

In this paper we derive the formulate for the Elzaki transform of partial derivatives and apply them in Solving five types of initial value problems. Our purpose here is to show the applicability of this interesting new transform and its effecting in solving such problems.

The concept of Differential transform was first introduced by Zhou [7, 8], who solved linear and nonlinear initial value problems in electric circuit analysis. Later, DTM was applied to many problems in the literature by several authors [10-13].

Let us consider the KdV equation with given initial condition [8].

$u_t + puu_x + qu_{xxx} = 0,$

$u(x,0) = f(x)$

where p and q are real constants, u_t and u_x denote partial derivatives with respect with space x and time t, and the nonlinear term uu_x tends to localize the wave, whereas the wave was spread out by dispersion. The formulation of solitons that have a single humped waves was define by delicate balance between uu_x and u_{xxx}. The displacement which describes how waves evolve under the competing but comparable effects of weak nonlinearity and weak dispersion is denoted by u(x,t).

Objectives

In order to achieve the main aim the following objectives will be carry out:

1 Introducing Adomian polynomial into Elzaki Transform method.

2 Applying the resulting scheme in 1 to solve nonlinear Kdv Equations which are initial value problem.

3 Finding the numerical solution of KdV equation using finite difference scheme.

4 To implement the discritize form or (system of equations) of KdV equation in Matlab.

Properties of Elzaki transform we need in this work

Elzaki transform is defined for function of exponential order [19].

Consider the functions in the set A define below

$$A=\left\{f(t): \exists M, c_1, c_2 > 0, |f(t)| < Me^{\frac{|t|}{c_j}},\ if\ t \in (-1)^j \times [0,\infty)\right\}$$

Note that for any given function in the set A defined above, the constant c1; c2 may be either finite or infinite, but M must be infinite.

According to Tarig M. Elzaki, Elzaki transform is defined as:

$$E[f(t)] = u^2\int_0^\infty f(ut)e^{-t}dt = T(u), \quad t \geq 0, u \in (c_1,c_2) \quad \text{(Or)}$$

$$E[f(t)\} = u\int_0^\infty f(t)\, e^{\frac{-t}{u}}dt = T(u), \quad t \geq 0, u \in (c_1,c_2)$$

Note that u in the above definition is used to factor t in the analysis of function f.

The next task is to state and prove the important theorem we are going to use in this work.

Let T(u) be the Elzaki transform of f(t) i.e, E[f(t)]= T(u), then:

(i) E[f'(t)]=$\frac{T(u)}{u} - uf(0)$

(ii) E[f''(t)= $\frac{T(u)}{u^2} - f(0) - uf'(0)$

(iii)E[$f^{(n)}$(t)]= $\frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0)$

Note that E[f(t)] = T(u) means that T(u) is the Elzaki transform of f(t), and f(t) is the inverse Elzaki transform of T(u). i.e,

f(t)=E^{-1}[T(u)]

Proof: (i) From the equation E[f'(t)] is given by:

E[f'(t)]= $u\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}$dt=T(u)

By integrating by part we have

$\int_0^{\infty} f'(t)\, e^{\frac{-t}{u}}$dt=$e^{\frac{-t}{u}} f(t)|_0^{\infty}$+$\frac{1}{u}\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}$dt,

=-f(0)+ $\frac{1}{u}\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}$dt

Which yields to

E[f'(t)]= $u\left[-f(0) + \frac{1}{u}\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}\text{dt}\right]$

=-f(0)+ $\frac{1}{u}\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}$dt

Recall that

T(u)= $u\int_0^{\infty} f(t)\, e^{\frac{-t}{u}}$dt

Therefore

E[f'(t)]=$\frac{T(u)}{u} - uf(0)$

(ii) Using equations again,we have

E[f''(t)]= $u\int_0^\infty f''(t)\, e^{\frac{-t}{u}}$dt

By integrating by part we have

E[f''(t)]= $u\left[-f'(0)+\frac{1}{u}\int_0^\infty f'(t)\, e^{\frac{-t}{u}}\text{dt}\right]$

E[f''(t)]= $-uf'(0)+\frac{E(f'(t))}{u}$

$=\frac{E(f'(t))}{u}-uf'(0)$

By considering the result from(i)

E[f''(t)= $\frac{T(u)}{u^2}-f(0)-uf'(0)$

(iii) For general form we are going to prove by induction

Given

E[$f^{(n)}$(t)]= $\frac{T(u)}{u^n}-\sum_{k=0}^{n-1}u^{2-n+k}f^{(k)}(0)$, for all n≥1

STEP 1: From equation when n=1, we have: E[f'(t)]=$\frac{T(u)}{u}-uf(0)$

$\frac{T(u)}{u}-uf(0)$ is clearly the same as $\frac{T(u)}{u^1}-u^{2-1+0}f^0(0)$, and the latter corresponds to equation when n=1.This shows that equation holds for n=1.

STEP 2: Assume that equation holds for n=N i.e,

E[$f^{(N)}$(t)]= $\frac{T(u)}{u^N}-\sum_{k=0}^{N-1}u^{2-N+k}f^{(k)}(0)$

We prove it equally hold for N+1, i.e,

E[$f^{(N+1)}$(t)]= $\frac{T(u)}{u^{N+1}}-\sum_{k=0}^{N}u^{2-(N+1)+k}f^{(k)}(0)$

To begin with, note that from STEP 1 above

E[$f^{(N+1)}$(t)]= E[$f^{(N)}$(t)']

$=\frac{E[f^{(N)}(t)]}{u}-uf^{(k)}(0)$

From equation

$E[f^{(N+1)}(t)]=\frac{T(u)}{u^{N+1}}-\sum_{k=0}^{N-1} u^{2-N+k-1} f^{(k)}(0) - uf^{(k)}(0)$

$=\frac{T(u)}{u^{N+1}}-\sum_{k=0}^{N} u^{2-(N+1)+k} f^{(k)}(0)$

The last equally corresponds to equation when n=N+1

Application of Elzaki transform and Adomian Polynomial for solving third order and fifth order KdV equations

In this section we want to incorporate the Adomian polynomial into the Elzaki transform to solve both third-order and fifth-order KdV Equations, and to generalized the method of solving any kind of these equations by using both Elzaki transform and Adomian polynomial. Let's take a look on our previous research work before solving KdV equations by Elzaki transform.

Application to third-order KdV Equations

Use the Elzaki transform method and Adomian polynomial to solve the following problems

Problem 1:

Consider the homogeneous KdV equation

$u_t+uu_x-uu_{xxx}+u_{xxxxx}+u_x^2=0$ →(1)

with initial conditions

$u(x,0)=e^x$

Let us rewrite Equation as:

$u_t=-[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]$

Applying Elzaki transform to both side

$E[u_t]=-E[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]$ →(2)

$E[u_t]=\frac{U(x,v)}{v}-vu(x,0)$

$\frac{U(x,v)}{v}-vu(x,0)=-E[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]$ →(3)

Applying the given initial conditions on Equation and simplifying, we obtain;

$U(x,v)=v^2e^x-vE[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]$ →(4)

Applying the inverse Elzaki transform,

$u(x,t)=E^{-1}[v^2e^x]-E^{-1}\{vE[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]\}$

The resulting expression is

$u(x,t)=e^x-E^{-1}\{vE[uu_x-uu_{xxx}+u_{xxxxx}+u_x^2]\}$→(5)

From $u_0=e^x$

The recursive relation is given as:

u_{n+1}=- $E^{-1}\left\{vE\left[An+\frac{\partial^5 u_n}{\partial x^5}-\left(\frac{\partial u_n}{\partial x}\right)^2\right]\right\}$ →(6)

Note that A_n is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}f[\sum_{i=0}^{\infty}\lambda^i u_i]_{\lambda=0}$ → (7)

Let the nonlinear term be represented by

$f(u)=u.\frac{\partial u}{\partial x}-u\frac{\partial^3 u}{\partial x^3}$ → (8)

By using Equation, we obtain;

$A_0= u_0\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]$

$A_1=u_1\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]+ u_0\left[\frac{\partial u_1}{\partial x}-\frac{\partial^3 u_1}{\partial x^3}\right]$

$A_2= u_2\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]+ u_1\left[\frac{\partial u_1}{\partial x}-\frac{\partial^3 u_1}{\partial x^3}\right]+ u_0\left[\frac{\partial u_2}{\partial x}-\frac{\partial^3 u_2}{\partial x^3}\right]$

From Equation (6)

When n=0, we have,

$u_1=- E^{-1}\left\{vE\left[A_0+\frac{\partial^5 u_0}{\partial x^5}-\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right\}$

$u_1=- E^{-1}\left\{vE\left[u_0\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]+\frac{\partial^5 u_0}{\partial x^5}-\left(\frac{\partial u_0}{\partial x}\right)^2\right]\right\}$

since $u_0= e^x$ we have,

$u_1=- E^{-1}\{vE[e^x-e^{2x}]\}$ → (9)

$u_1=-t[e^x-e^{2x}]$ → (10)

When n = 1, we have;

$u_2=-E^{-1}\left\{vE\left[A_0+\frac{\partial^5 u_1}{\partial x^5}-\left(\frac{\partial u_1}{\partial x}\right)^2\right]\right\}$

since $u_1= -t[e^x-e^{2x}]$ we have,

$u_2=-E^{-1}\left\{vE\left[u_1\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]+u_0\left[\frac{\partial u_1}{\partial x}-\frac{\partial^3 u_1}{\partial x^3}\right]+\frac{\partial^5(-t[e^x-e^{2x}])}{\partial x^5}-\left(\frac{\partial u_1}{\partial x}\right)^2\right]\right\}$ → (11)

$u_2=\frac{t^2}{2!}[e^x-e^{2x}]$

When n = 2, we have;

$u_3=-E^{-1}\left\{vE\left[A_2+\frac{\partial^5 u_2}{\partial x^5}-\left(\frac{\partial u_2}{\partial x}\right)^2\right]\right\}$

$u_3=-E^{-1}\left\{vE\left[u_2\left[\frac{\partial u_0}{\partial x}-\frac{\partial^3 u_0}{\partial x^3}\right]+u_1\left[\frac{\partial u_1}{\partial x}-\frac{\partial^3 u_1}{\partial x^3}\right]+u_0\left[\frac{\partial u_2}{\partial x}-\frac{\partial^3 u_2}{\partial x^3}\right]+\frac{\partial^5 u_2}{\partial x^5}-\left(\frac{\partial u_2}{\partial x}\right)^2\right]\right\}$ → (12)

$u_3=-\frac{t^3}{3!}[e^x-e^{2x}]$

The approximate series solution is,

$u(x,t)=u_0+u_1+u_2+\ldots$

u(x,t)= e^x-t$[e^x - e^{2x}] + \frac{t^2}{2!}[e^x - e^{2x}]$-$\frac{t^3}{3!}[e^x - e^{2x}]$+…

This can be written as

u(x,t)= $e^x - e^{2x}\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right]$

By using Taylor's series, the closed form solution will be as follows

e^{-t}=1-t+$\frac{t^2}{2!}$-$\frac{t^3}{3!}$+…

u(x,t)=)= $e^x . e^{-t}$

u(x,t)= e^{x-t}

Conclusion:

Generalizations of all existing Elzaki differentiation, integration and convolution theorems in the existing literature are demonstrated and so also generalizing all existing Elzaki shifting theorems. The Laplace – Elzaki duality (L E D) will be used to invoke a complex inverse Elzaki transform.

Moreover, the problems considered show that the Elzaki transform method and Adomian polynomial is a very powerful integral transform method in solving both third order and fifth order KdV equations. We have obtained the approximate analytical solutions of several third order and fifth order KdV equations using the combination of Elzaki transform method and Adomian polynomial which was meant for linearizing the nonlinear terms.Using this method make us to realise how potent this method is because all the problems considered yield exact solutions with few iterations. Solving any other nonlinear differential equations or ordinary is very easy by using this powerful method.

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