Minimum Irregularity of Totally Segregated Bicyclic Graphs

[1]Jorry T.F., [2]Parvathy K.S.

[1]Assistant Professor, [2]Associate Professor.
[1] Department of Mathematics,
[1]Mercy College , Palakkad, Kerala, India. [2]St. Mary's College , Thrissur, Kerala, India.

***Abstract :** The irregularity of a simple graph G = (V, E) is defined as irr(G) = $\sum_{uv\in E(G)} |deg_G(u) - deg_G|$, where $deg_G(u)$ denote the degree of a vertex u ∈ V(G). A graph in which any two adjacent vertices have distinct degrees is a Totally Segregated Graph. In this paper we determine minimum irregularity of three types of connected totally segregated bicyclic graph with n vertices. The extremal graphs are also presented.*

***IndexTerms** – Irregularity, totally segregated bicyclic graph, minimum irregularity.*

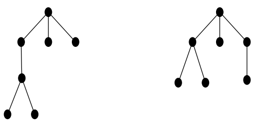
I. INTRODUCTION

In this paper we consider only simple undirected connected graphs. Let G = (V, E) be a graph of order n = |V(G)| and size m= |E(G)| . For u, v ∈ V(G), we denote the number of edges incident to v by $deg_G(v)$ or $d_G(v)$. Let P_n, C_n and S_n be the path, cycle, and star on n vertices respectively. To identify non-adjacent vertices x with y of a graph G is to replace these two vertices by a single vertex incident to all the edges which were incident in G to either x or y.

As well known, a graph whose vertices have equal degrees is said to be regular. Then a graph in which all the vertices do not have equal degrees can be viewed as somehow deviating from regularity. In mathematical literature, several measures of such 'irregularity' were proposed [4] [8] [7] [5]. One among them is the total irregularity of a graph ($irr_t(G)$) introduced by Abdo, Brandt and Dimitrov [2], which is defined as

$irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|$.

Another measure of 'irregularity' was put forward by Albertson [3]. Albertson defines the *imbalance* of an edge e = uv ∈ E(G) as $|deg_G(u) - deg_G(v)|$ and irregularity of G as irr(G) = $\sum_{uv\in E(G)} |deg_G(u) - deg_G(v)|$. The graph invarient irr(G) was sometimes referred to as Albertson index [8] or the third Zagreb index [6]. In this work, we use the terminology accepted by majority of the contemporary researchers [7] [9] [10] [1], according to which irr(G) is the irregularity of the graph G. In [3] Albertson presented upper bounds on irregularity for bipartite graphs, triangle free graphs and arbitrary graphs; also a sharp upper bound for trees. Some results about bipartite graphs given in [3] have been provided in [11]. Related to Albertson, [3] is the work of Hansen and Melot [9], who characterized the graphs with n vertices and m edges with maximal irregularity. In [1] Abdo, Cohen and Dimitrov presented an upper bound for irregularity for general graphs with n vertices. Note that the irregularity of a given graph is not completely determined by its degree sequence. Graphs with same degree sequence may have different irregularity. For example, (3, 3, 2, 1, 1, 1,1) is the degree sequence of the non-isomorphic graphs G_1 and G_2 in Figure 1. They have different irregularities (irr(G_1) = 10 and irr(G_2) = 8).



G_1 G_2

Figure 1.

In this paper, we focus on totally segregated bicyclic graphs on n vertices. For the sake of convenience totally segregated bicyclic graph is called a TSB graph. The notion of totally segregated graph is defined in [11]. A connected graph G is said to be totally segregated, if uv ∈ E(G), $deg_G(u) \neq deg_G(v)$. Sr. Jorry T.F. and Parvathy K.S. [12] studied a special case, by considering those graphs in which degrees of any two adjacent vertices are differed by a constant k ≠ 0 and these graphs are named as k-segregated. Minimum total irregularity of totally segregated tree is found in [13]. In [14] authors investigated the total irregularity of bicyclic graphs and characterized the graph with the maximal total irregularity among the bicyclic graphs on n vertices. In this paper, minimum irregularity of totally segregated bicyclic graphs of order n are determined and those extremal graphs are presented.

II. PRELIMINARIES

In [14] authors introduce different classes of bicyclic graphs. Here we refer those definitions.

A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. There are two basic bicyclic graphs: ∞- graph and θ- graph. A ∞ - graph denoted by ∞ (p,q,l) (see Figure 2), is obtained from two vertex-disjoint cycles C_p and C_q by connecting one vertex of C_p and one vertex of C_q with a path P_l of length l - 1 (in the case of l =1, identifying the above two vertices, (see Figure 3) where p, q ≥ 3 and l ≥1 ; and θ - graph, denoted by θ (p,q,l) (see Figure 4), is a graph on p + q - l vertices with the two cycles C_p and C_q having l common vertices, where p, q ≥ 3 and l ≥ 2.

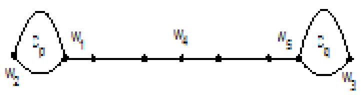


Figure 2

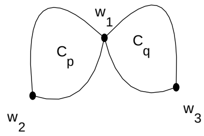


Figure 3

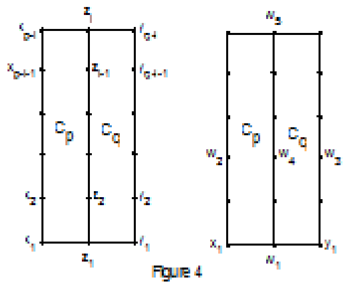


Figure 4

In Figure 2, let w_1 be the common vertex of P_l and C_p and let w_5 be the common vertex of P_l and C_q. Let $w_2 \in V(C_p) \setminus \{w_1\}$, $w_3 \in V(C_q) \setminus \{w_5\}$ and let $w_4 \in V(P_l) \setminus \{w_1, w_5\}$ if $l \geq 3$. In Figure 3, let $w_1 = V(C_p) \cap V(C_q)$; $w_2 \in V(C_p) \setminus V(C_q)$ and $w_3 \in V(C_q) \setminus V(C_p)$. In Figure 4, let $w_1 = z_1$, $w_2 \in \{x_1, x_2, \cdots, x_{p-1}\}$, $w_4 \in \{z_2, \cdots, z_{l-1}\}$ if $l \geq 3$, $w_3 \in \{y_1, y_2, \cdots, y_{q-1}\}$, and $w_5 = z_l$.

A rooted graph has one of its vertices, called the root, distinguished from the others. Root of the star S_n is its central vertex.

Let G_1 and G_2 be two graphs: $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. The graph $G = (G_1, v_1) * (G_2, v_2)$ denotes the graph resulting from identifying v_1 with v_2. Let $x \in V(\infty(p,q,l))$ and v be the root of the rooted tree T. Take $\infty(p,q,l,x* T) = (\infty(p,q,l,x)) * (T,v)$. In this case we say that tree T is attached to the graph $\infty(p, q, l)$ at x. For example, see Figure 5.

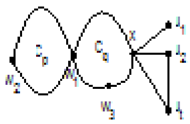


Figure 5

Remark 2.1

If a star S_2 is attached to $\infty(p,q,l)$ at w_2 the resulting graph G is denoted by $\infty(p, q, l, w_2 * S_2)$

Note that $\theta(p,q,l, w_1 * T) \cong \theta(p,q,l, w_5 * T)$ ■

A totally segregated bicyclic (TSB) graph is a bicyclic graph which is totally segregated. See Figure 6.

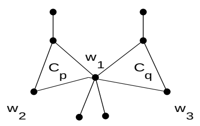


Figure 6

Observe that any bicyclic graph G is obtained from an ∞-graph or a θ-graph (possibly) by attaching trees to some of its vertices. If G is obtained from $\infty(p,q,l)$ by attaching trees to some of its vertices then we call G as bicyclic graph with basic bicycle $\infty(p,q,l)$ and if G is obtained from $\theta(p,q,l)$ by attaching trees to some of its vertices then we call G as bicyclic graph with basic bicycle $\theta(p,q,l)$. Obviously $\mathbf{B_n}$ consists of three types of bicyclic graphs of order n : first type, denoted by B_n, is the set of those graphs each of which is a bicyclic graph with basic bicycle $\infty(p,q,l)$, $p \geq 3$, $q \geq 3$, $l=1$ which is called ∞- bicyclic graph for convenience; second type, denoted by B^{+}_{n}, is the set of those graphs each of which is a bicyclic graph with basic bicycle $\infty(p,q,l)$, $p \geq 3$, $q \geq 3$, $l \geq 2$ which is called ∞^{+}- bicyclic graph ; third type, denoted by B^{++}_{n}, is the set of those graphs each of which is a bicyclic graph with basic bicycle $\theta(p,q,l)$, $p \geq 3$, $q \geq 3$, $l \geq 2$ which is called θ-bicyclic graph. Then, $\mathbf{B_n}$ = B_{n}$\cup B^{+}_{n} \cup B^{++}_{n}$. In this paper, we determine minimum irregularity of three types of totally segregated bicyclic graphs on n vertices.

III. TSB GRAPHS WITH MINIMUM IRREGYLARITY

3.1 Minimum Irregularity of Totally Segregated ∞- Bicyclic Graph on n Vertices

In the following theorem we find totally segregated ∞- bicyclic graphs on n vertices with minimum irregularity.

Theorem 3.1

Let B_n be the set of all totally segregated ∞- bicyclic graphs on n vertices, $(n \geq 7)$ and $B_n = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B$,

1. $B_1 = \{G \in B_n : n = 4k, k = 3,4,\cdots\cdots\}$
2. $B_2 = \{G \in B_n : n = 4k + 1, k = 3,4,5.\}$
3. $B_3 = \{G \in B_n : n = 4k + 1, k = 6,7,\cdots\cdots\}$
4. $B_4 = \{G \in B_n : n = 4k + 2, k = 3,4,\cdots\cdots\}$
5. $B_5 = \{G \in B_n : n = 4k+3, k = 3,4,\cdots\cdots\}$
6. $B = \{G \in B_n : n = 7,8,9,10,11.\}$

Then,

1. $\text{Min}\{irr(G)_{G\in B_1}\} = n + 2$
2. $\text{Min}\{irr(G)_{G\in B_2}\} = n + 3$
3. $\text{Min}\{irr(G)_{G\in B_3}\} = n + 1$
4. $\text{Min}\{irr(G)_{G\in B_4}\} = n + 2$
5. $\text{Min}\{irr(G)_{G\in B_5}\} = n + 1$
6. $\text{Min}\{irr(G)_{G\in B_n}\} = 12$, when n = 7, 8, 9 and $\text{Min}\{irr(G)_{G\in B_n}\} = 14$, when n = 10, 11.

Proof

Let B_n be the set of all totally segregated ∞- bicyclic graphs on n vertices. Any graph $G \in B_n$ has $n+1$ edges and $imb(e) \geq 1$ where $e \in E(G)$. Hence $irr(G) \geq n+1$.

Let $G \in B_1$. Since n is an even integer, G has odd number, (n+1), of edges . But irregularity of any graph is always even [3]. Hence at least one edge has an imbalance greater than one. Hence $irr(G) \geq n + 2$. In Figure 5, TSB graphs of B_1 with irregularity $n + 2$ are given.

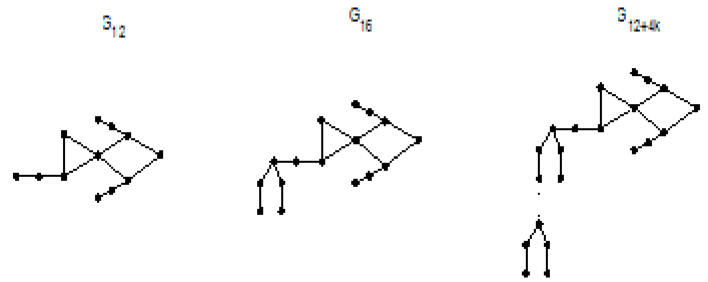


Figure 7

Let $G \in B_2$. In this case totally segregated ∞- bicyclic graphs with irregularity $n + 1$ or $n + 2$ does not exist.

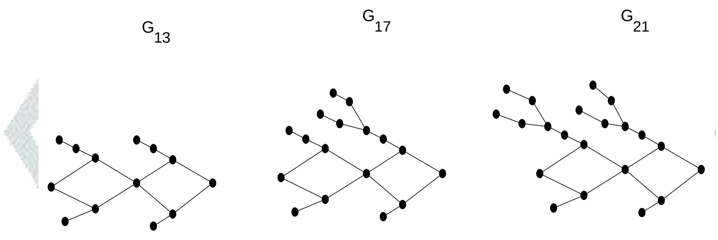


Figure 8

In Figure 8, TSB grphs of B_2 with irregularity $n + 3$ are presented.
Let $G \in B_3$. Since G is totally segregated, every edge of G has an imbalance of at least one. Hence $irr(G) \geq n + 1$.
TSB graphs of B_3 with irregularity $n + 1$ is presented in Figure 9.

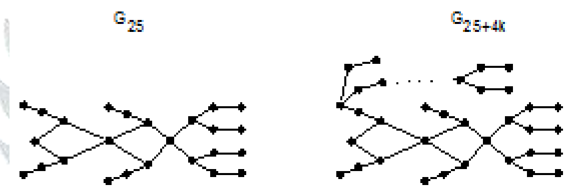


Figure 9

Let $G \in B_4$. In this case n is an even integer. Then the TSB graph G has odd number of edges (n+1). But irregularity of any graph is always even [3]. Hence $irr(G) \geq n + 2$. In Figure 10, TSB graphs of B_4 with irregularity $n + 2$ are given.

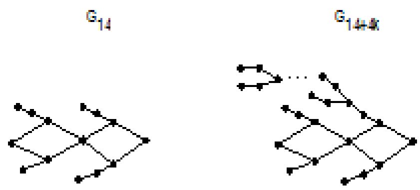


Figure 10

Let $G \in B_5$. G_{15} is a 1- segregated ∞- bicyclic graph on 15 vertices which is presented in figure 11. Identify the central vertex of P_5 with pendent vertex of G_{15}, whose adjacent vertex is the vertex with degree 2, to form G_{19}, 1-segregated ∞- bicyclic graph with 19 vertices. Repeat this process k times on G_{15} to get 1- segregated ∞- bicyclic graph on 4k + 3 vertices, k = 3,4,⋯ . Hence in this case $irr(G) = n + 1$. Totally segregated ∞- bicyclic graph with minimum irregularity on 4k+3 vertices, k = 3,4, ⋯ is given in Figure 11.

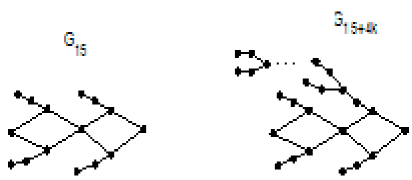


Figure 11

For n = 7, 8, 9, 10, 11 totally segregated ∞- bicyclic graph with minimum irregularity on n vertices is presented in Figure 12.

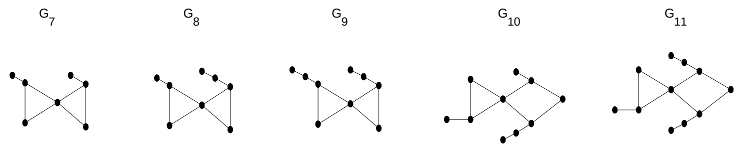


Figure 12

3.2 Minimum Irregularity of Totally Segregated ∞^+- Bicyclic Graph on n Vertices

In the following theorem we find totally segregated ∞^+- bicyclic graphs on n vertices with minimum irregularity.

Theorem 3.2

Let B^+_n be the set of all totally segregated ∞^+- bicyclic graphs on n vertices, $(n \geq 10)$

And $B^+_n = B^+_1 \cup B^+_2 \cup B^+_3 \cup B^+_4 \cup B^+$,

1. $B^+_1 = \{G \in B^+_n : n = 4k, k = 3,4,\cdots\cdots\}$
2. $B^+_2 = \{G \in B^+_n : n = 4k + 1, k = 3,4,5, \cdots\cdots.\}$
3. $B^+_3 = \{G \in B^+_n : n = 4k + 2, k = 4,5,\cdots\cdots\}$
4. $B^+_4 = \{G \in B^+_n : n = 4k+3, k = 4,5,\cdots\cdots\}$
5. $B^+ = \{G \in B+_n : n = 10,11,14,15.\}$

Then,

1. $\text{Min}\{irr(G)_{G\in B_1^+}\} = n + 2$
2. $\text{Min}\{irr(G)_{G\in B_2^+}\} = n + 1$
3. $\text{Min}\{irr(G)_{G\in B_3^+}\} = n + 2$
4. $\text{Min}\{irr(G)_{G\in B_4^+}\} = n + 1$
5. $\text{Min}\{irr(G)_{G\in B_n^+}\} = 20$ when n = 10, $\text{Min}\{irr(G)_{G\in B_n^+}\} = 14$ when n = 11 and $\text{Min}\{irr(G)_{G\in B_n^+}\}= 18$ when n=14, 15.

Proof

Let $G \in B^+_1$. Since G is bicyclic graph G has odd number of edges (n+1). But irregularity of any graph is always even [3]. Hence at least one edge has an imbalance greater than one. Hence $irr(G) \geq n+2$.

TSB graphs of B^+_1 with irregularity n + 2 are depicted in the Figure 13.

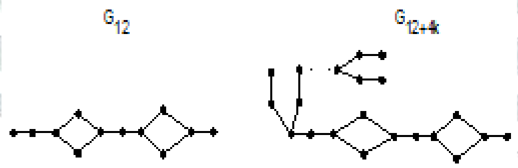


Figure 13

Let $G \in B^+_2$. Since G is a TSB graph, it has n+1 edges. Hence $irr(G) \geq n+1$. TSB graphs of B^+_2 with irregularity n+1 is presented in Figure 14.

G_{13} the 1 -segregated ∞^+- bicyclic graph with 13 vertices and 1- segregated graph of this type on 4k + 1 vertices, $k \geq 3$ is constructed from G_{13} is given in Figure 14.

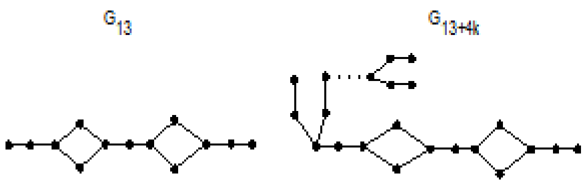


Figure 14

Let $G \in B^+_3$. In this case order of the graph G is even and hence it has odd number, (n+1), of edges. But irregularity of a graph is even [3]. Thus $irr(G) \geq n+2$. TSB graph $G \in B^+_3$ with irregularity n + 2 is presented in Figure 15.

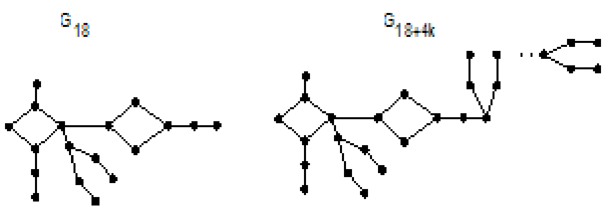


Figure 15

Let $G \in B^+_4$. Since G is TSB graph, it has n+1 edges. Hence $irr(G) \geq n+1$. TSB graphs of B^+_1 with irregularity n+1 is given in Figure 16.

G_{19} the 1- segregated ∞^{+} bicyclic graph with 19 vertices and 1- segregated ∞^{+}- bicyclic graph on 4k+3 vertices, k ≥ 4 is constructed from G_{19} is depicted in Figure 16.

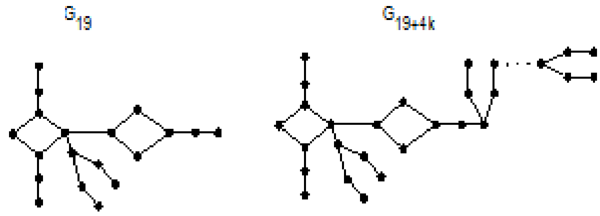


Figure 16

Let G ∈ B^{+}. Totally segregated ∞^{+}- bicyclic graph with minimum irregularity on n vertices, n = 10, 11, 14, 15, is presented in Figure 17.

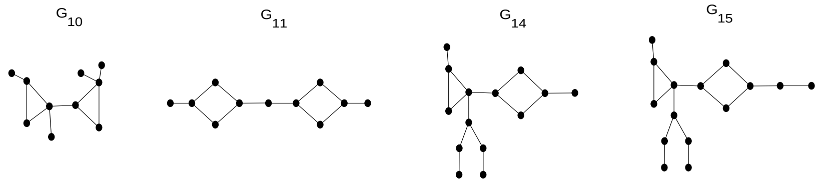


Figure 17

3.3 Minimum Irregularity of Totally Segregated θ- Bicyclic Graphs on n vertices

In the following theorem we find totally segregated θ- bicyclic graphs on n vertices with minimum irregularity.

Theorem 3.3

Let B^{++}_{n} be the set of all totally segregated θ- bicyclic graphs on n vertices, (n ≥ 5)

And $B^{++}_{n} = B^{++}_{1} \cup B^{++}_{2} \cup B^{++}_{3} \cup B^{++}_{4} \cup B^{++}$,

1. B^{++}_{1} = {G ∈ B^{++}_{n} : n = 4k, k = 3,4,⋯⋯}
2. B^{++}_{2} = {G ∈ B^{++}_{n} : n = 4k + 1, k = 3,4,5, ⋯⋯.}
3. B^{++}_{3} = {G ∈ B^{++}_{n} : n = 4k + 2, k = 3,4,5,⋯⋯}
4. B^{++}_{4} = {G ∈ B^{++}_{n} : n = 4k+3, k =3, 4,5,⋯⋯}
5. B^{++} = {G ∈ B^{++}_{n} : n = 5,6,7,8,9,10,11.}

Then,

6. Min$\{irr(G)_{G\in B_1^{++}}\}$ = n + 2
7. Min$\{irr(G)_{G\in B_2^{++}}\}$ = n + 1
8. Min$\{irr(G)_{G\in B_3^{++}}\}$ = n + 2
9. Min$\{irr(G)_{G\in B_4^{++}}\}$ = n + 1
10. Min$\{irr(G)_{G\in B_n^{++}}\}$ = 6 when n = 5, Min$\{irr(G)_{G\in B_n^{++}}\}$ = 8 when n = 6, 7, Min$\{irr(G)_{G\in B_n^{++}}\}$= 10 when n=8, 9. and Min$\{irr(G)_{G\in B_n^{++}}\}$= 14 when n = 10, 11.

Proof. Let G ∈ B^{++}_{1}. Since G is a bicyclic graph on n vertices, G has odd number, (n+1), of edges. But irregularity of any graph is always even [3]. Hence at least one edge has an imbalance greater than one. Hence irr(G) ≥ n+2 .

TSB graphs of B^{++}_{1} with irregularity n + 2 are depicted in Figure 18.

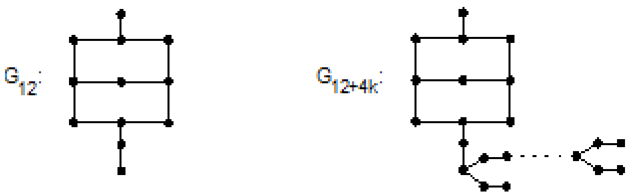


Figure 18

Let G ∈ B^{++}_{2}. Then irr(G) ≥ n+1. TSB-graphs of B^{++}_{2} with minimum irregularity n + 1 are presented in the Figure 19.

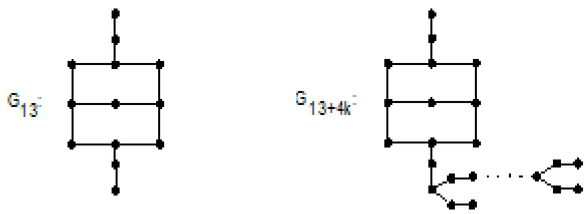


Figure 19

Let $G \in B^{++}_3$. Then $irr(G) \geq n+2$. TSB-graphs of B^{++}_3 with minimum irregularity $n + 2$ are presented in the Figure 20.

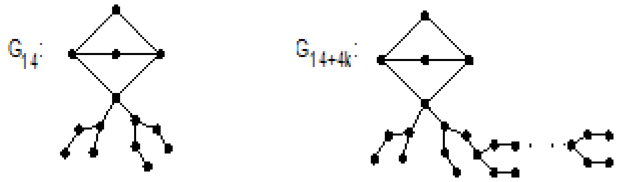


Figure 20

Let $G \in B^{++}_4$. Then $irr(G) \geq n+1$. TSB-graphs of B^{++}_4 with minimum irregularity $n+1$ are presented in the Figure 21.

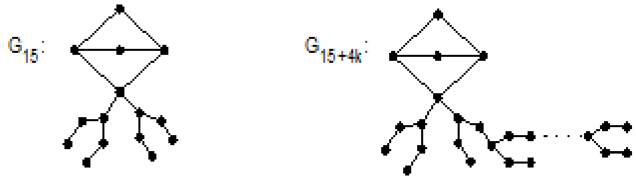


Figure 21

Let $G \in B^{++}$. Totally segregated θ- bicyclic graph with minimum irregularity are presented in Figure 22.

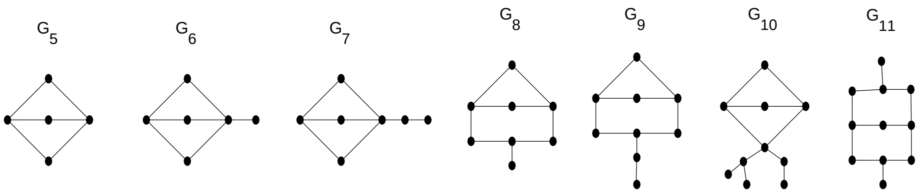


Figure 22

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