The Fermat Meet and Reciprocal Fermat Meet Matrices on A sets

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Abstract

Fermat meet and Reciprocal Fermat meet matrices on A sets are considered and their determinants, inverses and the matrices are expressed in terms of A-sets.

Key words : Meet Matrices, Fermat Meet Matrices, Reciprocal Meet Matrices, a- Set, A-Set

1 Introduction

Let $S = \{x_1, x_2, ..., x_n\}$ be a set of distinct positive integesr ,and let f be an arithmetical function. Then $n \times n$ matrix (S) whose i,j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S[1, 2, 4, 5].The set S is said to be *factor-closed* if it contains every divisor of any element of S,and the set S is said to be *GCD-closed* if it contains the greatest common divisor of any two elements of S[2, 3, 7].
Let $(P, \wedge)$ be a meet-semilattice and let $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ be a subset of P. Then S is an A-set if $A = \{x_i \wedge x_j \ / \ x_i \neq x_j\}$ is a chain. For example, chains and a-sets (with $A = \{a\}$) are known trivial A-sets. The meet matrix $(S)_f$ on S with respect to a function $f : P \to \mathbf{C}$ is defined as $((S)_f)_{ij} = f(x_i \wedge x_j)$.

If $f(x_i \wedge x_j) = 2^{2^{x_i \wedge x_j}} + 1$ then the n × n meet matrix obtained is called the Fermat meet matrix on S.

If $f(x_i \wedge x_j) = \frac{1}{2^{2^{x_i \wedge x_j}}+1}$ then the n × n meet matrix obtained is called the Reciprocal Fermat meet matrix on S.

We say that S is an ***A*-set** if the set $A = \{x_i \wedge x_j / x_i \neq x_j\}$ is a chain (an A-set need not be meet-closed). For example, chains and a-sets (with $A = \{a\}$ are known trivial A sets. Since the method, presented in [10], adapted to A-sets might not be sufficiently effective, we give a new structure theorem for $(S)_f$ where S is an A-set. One of its features is that it supports recursive function calls.
By the structure theorem we obtain a recursive formula for $\det(S)_f$ and for $(S_f)^{-1}$ on A-sets. By dissolving the recursion on certain sets we also obtain e.g. the known explicit determinant and inverse formulae on chains and a-sets.

Note that $(\mathbf{Z}+,|) = (\mathbf{Z}+, \gcd, \mathrm{lcm})$ is a locally finite lattice, where | is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i , x_j))$ and therefore the results in this paper also hold for GCD. For general accounts of GCD matrices, see [11]. Meet matrices are also generalizations of GCUD matrices, the unitary analogies of GCD matrices, see [12]. Thus the results also hold for GCUD matrices.

DEFINITIONS

Definition 2.1 *The binary operation* $\sqcap$ *is defined by*

$$S_1 \sqcap S_2 = [x \wedge y \,/\, x \in S_1, y \in S_2, x \neq y] \tag{2.1}$$

where S_1 *and* S_2 *are nonempty subsets of* P. *Let* S *be a subset of* P *and let* $a \in P$. *If* $S \sqcap S = \{a\}$, *then the set* S *is said to be an* ***a-set****.*

Definition 2.2 *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be a subset of* P *with* $x_i < x_j \Rightarrow i < j$ *and let* $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ *be a multichain (i.e. a chain where duplicates are allowed) with* $a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_{n-1}$. *The set* S *is said to be an* ***A-set*** *if* $\{x_k\} \sqcap \{x_{k+1}, \ldots\ldots, x_n\} = \{a_k\}$ *for all* $k = 1, 2, \ldots\ldots, n-1$.

Every chain $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ is an A-set with $A = S \setminus \{x_n\}$ and every a-set is always an A-set with $A = \{a\}$.

Definition 2.3 *Let* f *be a complex-valued function on* P. *Then the* $n \times n$ *matrix* $(S)_f$, *where* $((S)_f)_{ij} = f(x_i \wedge x_j)$, *is called the meet matrix on* S *with respect to* f. *Also the* $n \times n$ *matrix* $(S)_f$ *,where* $((S)_f)_{ij} = f(x_i \wedge x_j) = 2^{2^{x_i \wedge x_j}} + 1$ is called the Fermat meet matrix on S.

Definition 2.4 *Let* f *be a complex-valued function on* P. *Also the* $n \times n$ *matrix* $(S)_f$ *,where* $((S)_f)_{ij} = f(x_i \wedge x_j) = \frac{1}{2^{2^{x_i \wedge x_j}} + 1}$ is called the Reciprocal Fermat meet matrix on S.

In what follows, let $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ always be a finite subset of P with $x_i < x_j \Rightarrow i < j$. Let also $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ with $a_i < a_j \Rightarrow i < j$. Note that S has always n distinct elements, but it is possible that the set A is a multiset. Let f be a complex-valued function on P.

3 FERMAT AND RECIPROCAL FERMAT MEET MATRICES ON *A*-SETS

3.1 Structure Theorem

Theorem 3.1 (Structure Theorem) *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \ldots\ldots, f_n$ *denote the functions on* P *defined by* $f_1 = f$ *and*

$$f_{k+1}(x) = f_k(x) - \frac{f_k(a_k)^2}{f_k(x_k)} \tag{3.1}$$

for $k = 1, 2, \ldots\ldots\ldots, n - 1$.

Then

$$(S)_f = M^T DM, \tag{3.2}$$

where $D = \mathrm{diag}\,(f_1(x_1), f_2(x_2), \ldots\ldots\ldots, f_n(x_n))$ *and* M *is the* $n \times n$ *upper triangular matrix with* 1*'s on its main diagonal, and further*

$$(M)_{ij} = \frac{f_i(a_i)}{f_i(x_i)} \tag{3.3}$$

for all $i < j$. *(Note that* $f_1, \ldots\ldots\ldots, f_n$ *exist if and only if* $(f_k(x_k) = 0, a_k \neq x_k) \Rightarrow f_k(a_k) = 0$

holds for all $k = 1, 2, \ldots, n-1$. *In the case* $f_k(a_k) = f_k(x_k) = 0$ *we can write e.g.* $(M)_{kj} = 0$ *for all* $k < j$.

Proof : Let $i < j$. Then

$$(M^{T}DM)_{ij} = \sum_{k=1}^{n} (M)_{ki}(D)_{kk}(M)_{kj} = f_i(a_i) + \sum_{k=1}^{i-1} \frac{f_k(a_k)^2}{f_k(x_k)} \tag{3.4}$$

$$= f_i(a_i) + \sum_{k=1}^{i-1} (f_k(a_i) - f_{k+1}(a_i)) = f_1(a_i) = f(x_i \wedge x_j).$$

The case $i = j$ is similar, we only replace every a_i with x_i in (3.4). Since $M^{T}DM$ is symmetric, we do not need to treat the case $i > j$.

3.2 Determinant of Fermat Meet and Reciprocal Fermat Meet matrices on *A*-sets

By Structure Theorem we obtain a new recursive formula for $\det(S)_f$ on A-sets.

Theorem 3.2 *Let* $S = \{ x_1, x_2, \ldots\ldots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \ldots\ldots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \ldots, f_n$ *be the functions defined in* (3.1). *Then*

$$\det (S)_f = f_1(x_1)f_2(x_2)\ldots\ldots f_n(x_n), \tag{3.5}$$

By Theorem 3.2 we obtain a known explicit formula for $\det(S)_f$ on chains presented in [7, Corollary 3] and [13, Corollary 1].

Corollary 3.1 *If* $S = \{ x_1, x_2, \ldots\ldots x_n \}$ *is a chain, then*

$$\operatorname{Det} (S)_f = f(x_1) \prod_{k=2}^{n}(f(x_k) - f(x_{k-1})) \tag{3.6}$$

Proof : By Theorem 3.2 we have
$\det(S)_f = f_1(x_1)f_2(x_2)\ldots\ldots f_n(x_n)$, where $f_1 = f$ and
$f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all $k = 1, 2, \ldots\ldots, n-1$. This completes the proof.

By Theorem 3.2 we also obtain a known explicit formula for $\det(S)_f$ on a-sets. This formula has been presented (with different notation) in [13, Corollary of Theorem 3] and [10,Corollaries 5.1 and 5.2], and also in [16, Theorem 3] in number-theoretic setting.
The case $f(a) = 0$ is trivial, since then $(S)_f = \operatorname{diag}(f(x_1), f(x_2), \ldots\ldots, f(x_n))$ and $\det(S)_f = f(x_1)f(x_2)\ldots\ldots f(x_n)$.

Corollary 3.2 *Let* $S = \{ x_1, x_2, \ldots\ldots x_n \}$ *be an a-set, where* $f(a) \neq 0$. *If* $a \in S$ *(i.e.* $a = x_1$*), Then* $\det(S)_f = f(a)(f(x_2) - f(a))\ldots (f(x_n) - f(a))$. (3.7)

If $a \notin S$*, then*

$$\det(S)_f = \sum_{k=1}^{n} \frac{f(a)(f(x_1) - f(a))\ldots\ldots(f(x_n) - f(a))}{f(x_k) - f(a)} + (f(x_1) - f(a))\ldots (f(x_n) - f(a)). \tag{3.8}$$

Example 3.1 Fermat matrix

Let $(P, \leq\cdot) = (\mathbf{Z}+,|)$ and $S = \{1,2,3\}$.

Then S=$\begin{bmatrix} 2^{2^1}+1 & 2^{2^1}+1 & 2^{2^1}+1 \\ 2^{2^1}+1 & 2^{2^2}+1 & 2^{2^1}+1 \\ 2^{2^1}+1 & 2^{2^1}+1 & 2^{2^3}+1 \end{bmatrix}$ Since S is an A-set with the chain A = {1,1} by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(1)^2/f_1(1)$ and $f_3(x) = f_2(x) - f_2(1)^2/f_2(2)$. and. Let $f(x) = 2^{2^x}+1$. Then $f_1(x) = 2^{2^x}+1$, $f_2(x) = 2^{2^x}-4$, $f_3(x) = 2^{2^x}-4$

and by Theorem 3.1 $(S)_f = M^TDM$, where D=diag(5,12,252)and M = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and by Theorem 3.2 we have $\det(S)_f = f_1(1)f_2(2)f_3(3) = 5(12)(252) = 15120$.

Example 3.1 Reciprocal Fermat matrix

Let $(P, \leq) = (\mathbf{Z}+,|)$ and S ={1,2,3} .

Then S=$\begin{bmatrix} \frac{1}{2^{2^1}+1} & \frac{1}{2^{2^1}+1} & \frac{1}{2^{2^1}+1} \\ \frac{1}{2^{2^1}+1} & \frac{1}{2^{2^2}+1} & \frac{1}{2^{2^1}+1} \\ \frac{1}{2^{2^1}+1} & \frac{1}{2^{2^1}+1} & \frac{1}{2^{2^3}+1} \end{bmatrix}$ Since S is an A-set with the chain A = {1,1} by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(1)^2/f_1(1)$ and $f_3(x) = f_2(x) - f_2(1)^2/f_2(2)$.

Let $f(x) = 1/(2^{2^x}+1)$. Then $f_1(x) = \frac{1}{2^{2^x}+1}$, $f_2(x) = \frac{1}{2^{2^x}+1} - \frac{1}{5}$, $f_3(x) = \frac{1}{2^{2^x}+1} - \frac{1}{5}$

and by Theorem 3.1 $(S)_f = M^TDM$, where D=diag(1/5,-12/85,-252/1285)and M = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and by Theorem 3.2 we have $\det(S)_f = f_1(1)f_2(2)f_3(3)$ =(1/5)(-12/85)(-252/1285) = 3024/5,46,125.

3.3 Inverse of Fermat and Reciprocal Fermat meet matrices on *A*-sets

By Structure Theorem we obtain a new recursive formula for $(S_f)^{-1}$ on A-sets.

Theorem 3.3 *Let* $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ *be an A-set, where* $A = \{a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1}\}$ *is a multichain. Let* $f_1, f_2, \ldots\ldots, f_n$ *be the functions defined in* (3.1), *where* $f_i(x_i) \neq 0$ *for* $i = 1,2,,\ldots,n$. *Then* $(S)_f$ *is invertible and* $(S_f)^{-1} = N\triangle N^T$ (3.9)

where $\triangle = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ *and N is the n×n upper triangular matrix with* 1*'s on its main diagonal, and further*

$$(N)_{ij} = -\frac{f_i(a_i)}{f_i(x_i)} \prod_{k=i+1}^{j-1}\left(1 - \frac{f_k(a_k)}{f_k(x_k)}\right) \qquad (3.10)$$

for all i < j.

Proof: By Structure Theorem $(S)_f = M^TDM$, where M is the matrix defined in (3.3) and $D = \mathrm{diag}(f_1(x_1), f_2(x_2), \ldots\ldots, f_n(x_n))$. Therefore $(S_f)^{-1} = N\triangle N^T$,

where $D^{-1} = \mathrm{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ and $M^{-1} = N$ is the $n \times n$ upper triangular matrix in 3.10.

Example 3.1.1

S is considered the same as in Example 3.1 and by $(S_f)^{-1} = N\triangle N^T$,

then for Fermat matrix

$\triangle$=diag (1/5,1/12,1/252)) , N= M^{-1}, N= $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$(S_f)^{-1} = \begin{bmatrix} \frac{362}{1260} & -\frac{1}{12} & -\frac{1}{252} \\ -\frac{1}{12} & \frac{1}{12} & 0 \\ -\frac{1}{252} & 0 & \frac{1}{252} \end{bmatrix}$$

And for Reciprocal Fermat matrix,

$\triangle$=diag (5,-85/12,-1285/252)) , N= M^{-1}, N= $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$(S_f)^{-1} = \begin{bmatrix} -\frac{905}{126} & \frac{85}{12} & \frac{1285}{252} \\ \frac{85}{12} & -\frac{85}{12} & 0 \\ \frac{1285}{252} & 0 & -\frac{1285}{252} \end{bmatrix}$$

Corollary 3.3 *Let S* = { $x_1, x_2,\ldots\ldots\ldots\ldots x_n$ } *be an a-set, where f(a)* $\neq$0 *and f(x$_k$)* $\neq$ *f(a) for all k* = 2,……., *n. If a* $\in$ *S (i.e. a* = x_1*), then* $(S)_f$ *is invertible and*

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(a)} + \sum_{k=2}^{n} \frac{1}{f(x_k)-f(a)} & if\ i = j = 1, \\ \frac{1}{f(x_k)-f(a)} & if\ 1 < i = j, \\ \frac{1}{f(a)-f(x_k)} & if\ 1 = i < j = k\ or\ 1 = j < i = k \\ 0 & otherwise \end{cases} \tag{3.11}$$

If a $\notin$ *S and further f(x$_1$)* $\neq$*f(a) and* $\frac{1}{f(a)} \neq \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}$ *, then* $(S)_f$ *is invertible and*

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(x_k)-f(a)} - \frac{1}{[f(x_k)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i = j, \\ \frac{1}{[f(x_k)-f(a)][f(x_k)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i \neq j. \end{cases} \tag{3.12}$$

CONCLUSION:

In this paper we prove by examples that the Fermat Meet and Reciprocal Fermat matrices on A sets satisfies structure theorem and calculate the determinant and inverse of these matrices through results based on A sets.

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