A STUDY ON PASCAL'S TRIANGLE

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Abstract: The Pascal's Triangle is a mathematically rich concept. The paper focuses on the properties of Pascal's Triangle: Powers of 2, Exponential form of 11, Binomial coefficients, Combinations, Catalan numbers, Hemachandra -Fibonacci numbers and its applications, Golden Ratio, Sierpinski triangle and Pascal's Flower.

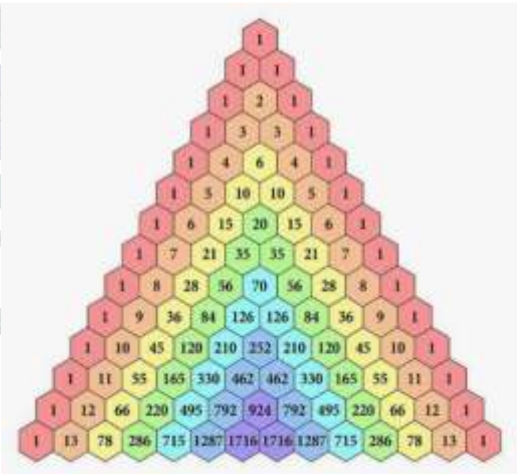
IndexTerms **- Binomial Coefficients, Combinations, Hemachandra-Fibonacci number, Sierpinski triangle**

I. INTRODUCTION

A perspicuous representation of structures makes the language of mathematics more comprehensible, such as the Pascal's Triangle. It is named after the French mathematician and physicist Blaise Pascal, who described this in 1653, even though many other mathematicians from India, Persia, China, Germany and Italy discovered this structure centuries before him.

Indian mathematicians called this structure as the Staircase of Mount Meru, Chinese mathematicians called it as the Yang Hui's triangle and in Iran it is known as the Khayyam triangle.

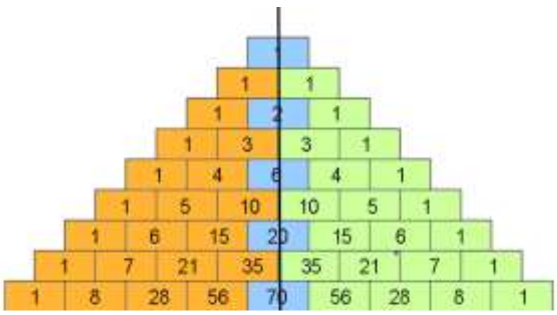
The Pascal's triangle is defined as a triangular array of binomial coefficients. The first row is the zeroth row (represented as n=0) which has a unique entry of one. The subsequent terms in each row of the triangle are obtained as the sum of the adjacent entries directly above it, considering the vacant entries as having magnitude zero. In the n^{th} row, the first entry is in the zeroth column, the second entry is in the first column…and the last entry is in the n^{th} column.



II. PROPERTIES OF PASCAL'S TRIANGLE

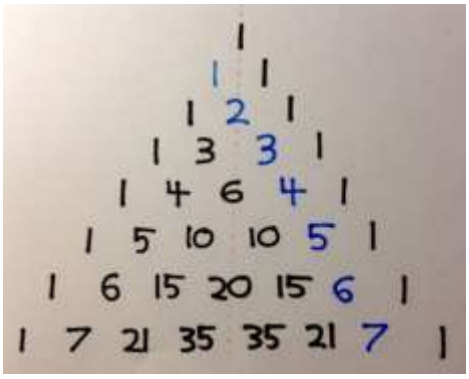
The Pascal's triangle follows a variety of patterns and finds its use in algebra and combinatorics. Some of the properties are described below:

1. BILATERAL SYMMETRY



A vertical line drawn through the middle of the Pascal's triangle divides the triangle into two identical halves like mirror images, excluding the entries that fall on the vertical line.

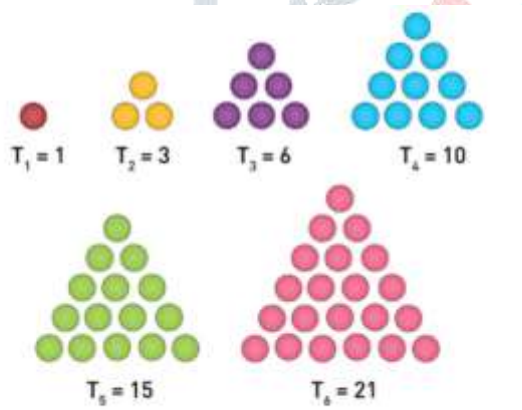
2. NATURAL NUMBERS

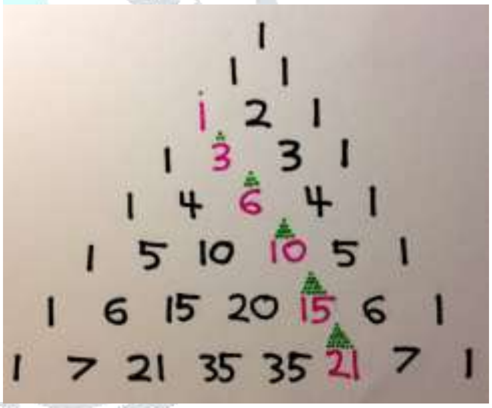


The second diagonal of the Pascal's triangle represents the natural numbers, commonly known as the counting numbers, symmetrically.

3. TRIANGULAR NUMBERS

A Triangular number is a figurative number which represents the number of dots that can be uniformly arranged in an equilateral triangle.
The sequence is 1,3,6,10,15…
The general formula of the sequence is n(n+1) /2, where n=1,2, 3…
i.e. the n^{th} triangular number is the sum of the first n natural numbers.

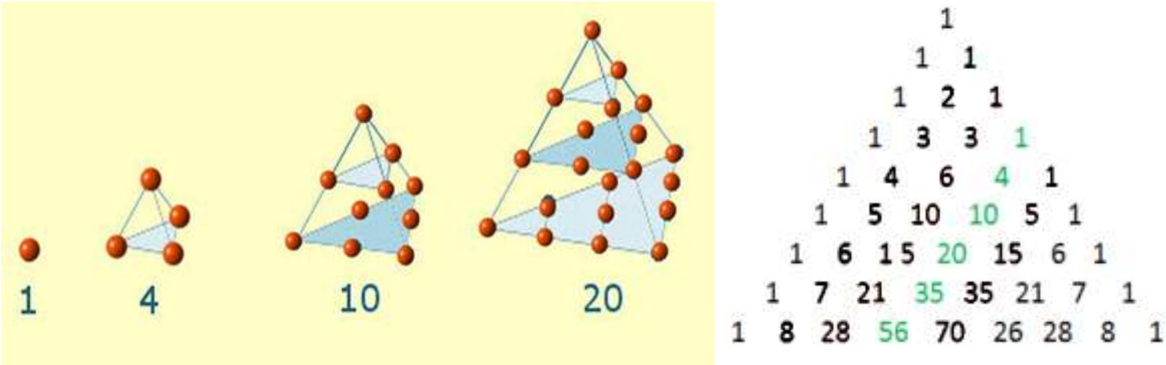




These numbers are found on the third diagonal of the Pascal's triangle symmetrically.

4. TETRAHEDRAL NUMBERS

A Tetrahedral number is a figurative number which represents the number of dots that can be uniformly arranged in a regular tetrahedron, which is a triangular base pyramid.
The sequence is 1,4,10,20…
The general formula of the sequence is n(n+1) (n+2)/6 where n=1,2, 3...



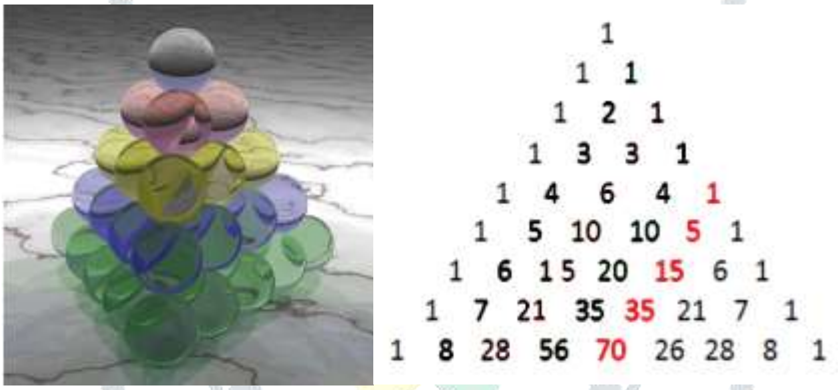
These numbers are found on the fourth diagonal of the Pascal's triangle symmetrically.

5. PENTATOPE NUMBERS

A pentatope number is a figurative number which represents the number of dots that can be uniformly arranged to form a pentatope, which is a four-dimensional analogue of a solid tetrahedron.
The sequence is 1,5,15,35…

The general formula of the sequence is (n+3) Tn/4=n(n+1) (n+2) (n+3)/24

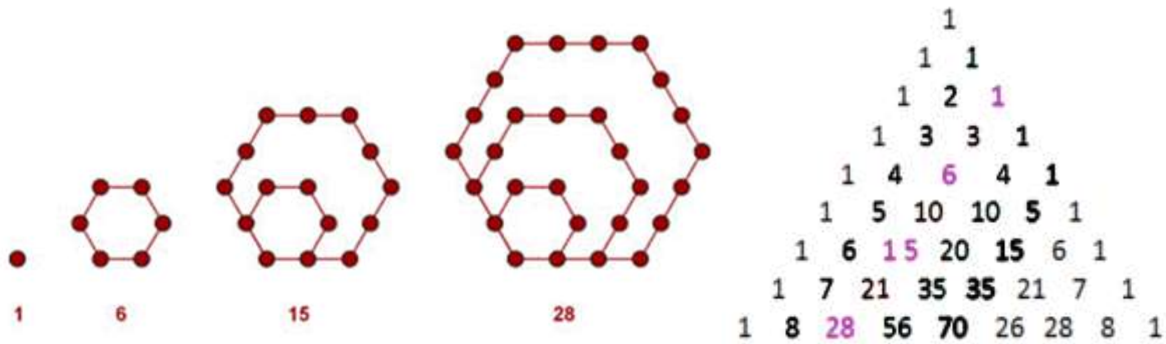
where T_n is the n^{th} tetrahedral number and n=1,2,3…



These numbers are found on the fifth diagonal of the Pascal's triangle symmetrically.

6. HEXAGONAL NUMBERS

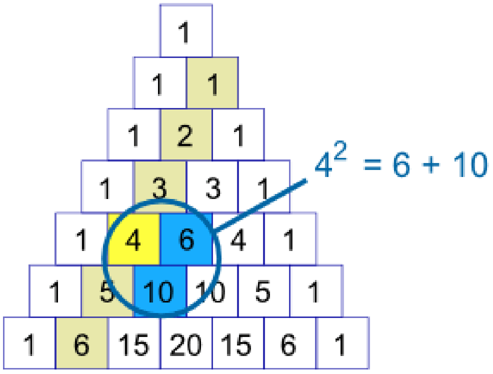
A hexagonal number is a figurative number which represents the number of dots that can be uniformly arranged in a regular hexagon.
The sequence is 1,6,15,28…
The general formula of the sequence is n(2n-1) where n=1,2,3…



These numbers are found on the third diagonal of the Pascal's triangle in alternate positions symmetrically.

7. SQUARE NUMBERS

The squares of the numbers are obtained from the Pascal's triangle.

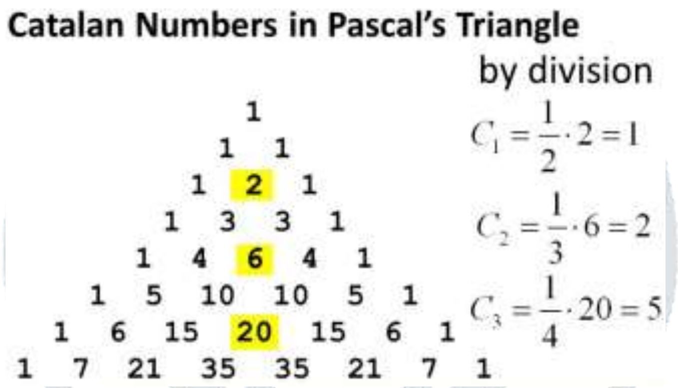


The square of an entry in the second diagonal equals the sum of the entries in the third diagonal directly adjacent to it.

8. CATALAN NUMBERS

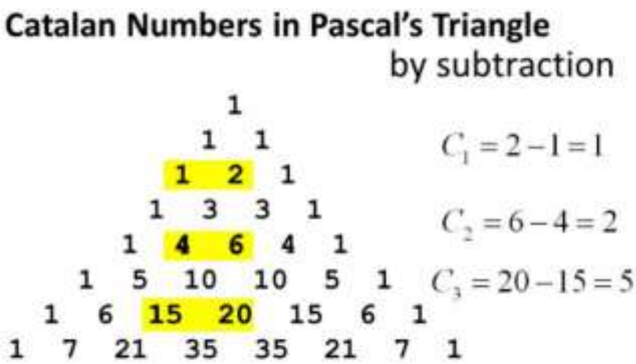
Catalan numbers are sequence of natural numbers that occur in various counting problems in combinatorial mathematics.
The sequence is 1,1,2,5,14…
The general formula of the sequence is (2n)! /(n+1)! n! where n=0,1,2, 3...
The Catalan numbers can be obtained from the Pascal's triangle in different ways:

8.1 BY DIVISION



The middle terms of the even numbered rows divided by successive natural numbers generates the sequence of the Catalan numbers.
For example, the middle term of the zeroth row which is 1 when divided by the first natural number 1equals 1.
Similarly, the middle term of the second row which is 2 when divided by 2 equals 1.
Thus, following this pattern, the successive terms of the Catalan numbers are obtained from Pascal's Triangle by division.

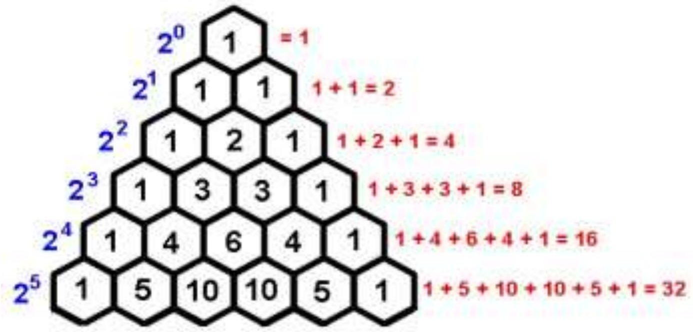
8.2 BY SUBTRACTION



The difference between the middle terms of the even numbered rows and the adjacent entries in the same row generates the sequence of the Catalan numbers.
For example,the middle term of the second row which is 2 subtracted by the adjacent entry 1 equals 1.

Similarly, the middle term of the fourth row which is 6 subtracted by the adjacent entry 4 equals 2.
Thus, following this pattern, the successive terms of the Catalan numbers are obtained.

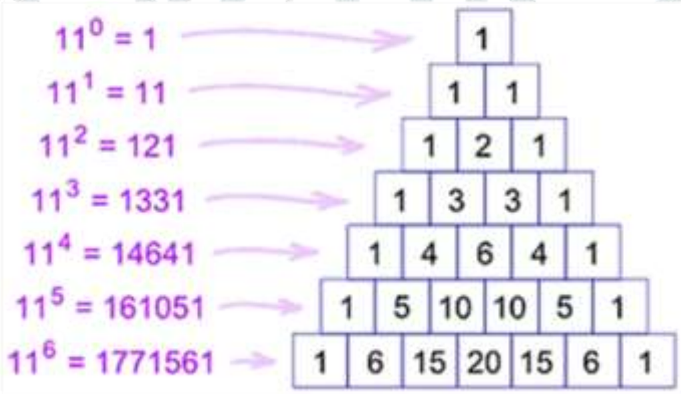
9. POWERS OF 2



The sum of the entries in each row is a power of 2.

For example, in the first row, the sum of the entries is $1+1=2=2^1$
i.e. sum of the entries in the n^{th} row = 2^n, where n=0,1,2,3… and hence sum forms the sequence 1,2,4,8…

10. EXPONENTIAL FORM OF 11



The entries in each row represents the powers of 11.

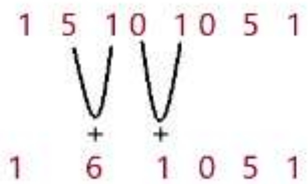
For example,11^1=11, which are the entries of the first row of the Pascal's triangle.

This pattern is used to obtain the powers of 11 till the fourth power.

Subsequent powers of 11 are obtained from the Pascal's triangle in a slightly different manner, by carrying over if an entry is not a single digit number.

For example, to obtain 11^5,we take the first entry of the fifth row, which is 1, then we take the sum of the next two digits,5+1=6,again we take the sum of the next two digits,0+1=1,and the rest of digits are taken as it is in that order,i.e.051.
Thus, we obtain 11^5=161051.

This pattern is illustrated below:



The rest of the powers of 11 are obtained from the Pascal's Triangle by following this pattern.

11. BINOMIAL COEFFICIENTS

The binomial coefficients in the expansion of a binomial equation in two variables, say x and y are obtained directly from the Pascal's triangle.

For example, $(x+y)^0=1$, is the entry of zeroth row.
$(x+y)^1=1x+1y$, here the coefficients of x and y, which is 1 and 1, are the entries of first row and the rest of the expansions follows this pattern as illustrated below:

$$(x+y)^0=1 \quad \text{—— 0th row}$$
$$(x+y)^1=1x+1y \quad \text{—— 1st row}$$
$$(x+y)^2=1x^2+2xy+1y^2 \quad \text{—— 2nd row}$$
$$(x+y)^3=1x^3+3x^2y+3xy^2+1y^2 \quad \text{—— 3rd row}$$
$$(x+y)^4=1x^4+4x^3y+6x^2y^2+4xy^3+1y^4 \quad \text{—— 4th row}$$
$$(x+y)^5=1x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+1y^5 \quad \text{—— 5th row}$$

Therefore, the binomial coefficients in the expansion of two variables say x and y raised to a power "n" are obtained as the entries of the n^{th} row of the Pascal's triangle.

12. PROBABILITY

The Pascal's triangle represents the ways in which heads and tails combine when coins are tossed thus giving 'the odds' of any combination.
For example, consider the case of the experiment as the number of heads obtained as the outcome.

When a coin is tossed there are two outcomes, either a head or a tail (no head). This is represented by the entries of the first row which add up to 2, thus giving ½ as the probability for no head and ½ as the probability for one head.

Table 1

Tosses	Possible Results (Grouped)	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1, 2, 1
3	HHH HHT, HTH, THH HTT, THT, TTH TTT	1, 3, 3, 1
4	HHHH HHHT, HHTH, HTHH, THHH HHTT, HTHT, HTTH, THHT, THTH, TTHH HTTT, THTT, TTHT, TTTH TTTT	1, 4, 6, 4, 1
	... etc ...	

Thus, the number of combinations of heads and tails are obtained from the row corresponding to the number of tosses of the coin.

13. COMBINATIONS

Combination is the selection of 'r' objects that can be formed out of 'n' total objects in which the order is not important. The general formula is

$$nCr = \frac{n!}{r!(n-r)!}$$

Each entry in the Pascal's triangle follows this formula where n is the row number and r is the column number as given below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$
\begin{array}{cl}
{}^{0}C_{0} & n=0\\
{}^{1}C_{0}\ {}^{1}C_{1} & n=1\\
{}^{2}C_{0}\ {}^{2}C_{1}\ {}^{2}C_{2} & n=2\\
{}^{3}C_{0}\ {}^{3}C_{1}\ {}^{3}C_{2}\ {}^{3}C_{3} & n=3\\
{}^{4}C_{0}\ {}^{4}C_{1}\ {}^{4}C_{2}\ {}^{4}C_{3}\ {}^{4}C_{4} & n=4\\
{}^{5}C_{0}\ {}^{5}C_{1}\ {}^{5}C_{2}\ {}^{5}C_{3}\ {}^{5}C_{4}\ {}^{5}C_{5} & n=5
\end{array}
$$

Thus, we can obtain the ways in which selections can be formed out of a total number of ‘n’ things from the entries of the n^{th} row.

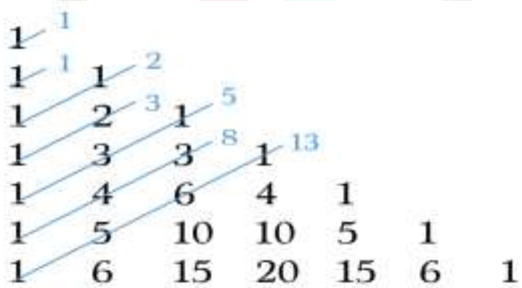
14. HEMACHANDRA-FIBONACCI NUMBERS

The Hemachandra-Fibonacci numbers were earlier named after the Italian mathematician Fibonacci who described these in 1202.But recently it was discovered that Acharya Hemachandra, an Indian Jain scholar described this sequence in 1150, almost half a century before Fibonacci and hence it has been renamed as Hemachandra-Fibonacci numbers.

The sequence is 1,1,2,3,5,8,13…and each term in this sequence is the sum of the preceding two terms as given below:

1+1=2
1+2=3
2+3=5,..

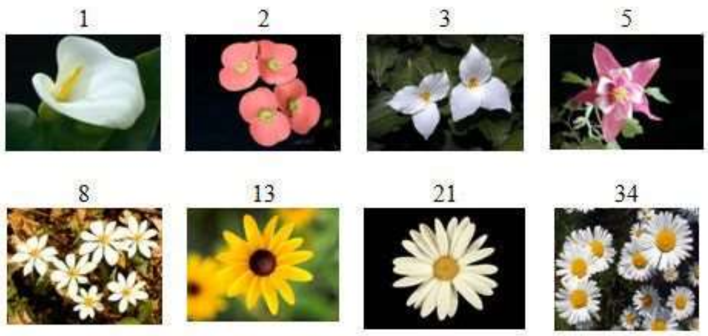
When the entries of the Pascal’s triangle are left aligned, the sum of the entries in the n^{th} diagonal is equal to the n^{th} term in the sequence of Hemachandra-Fibonacci numbers as illustrated below:



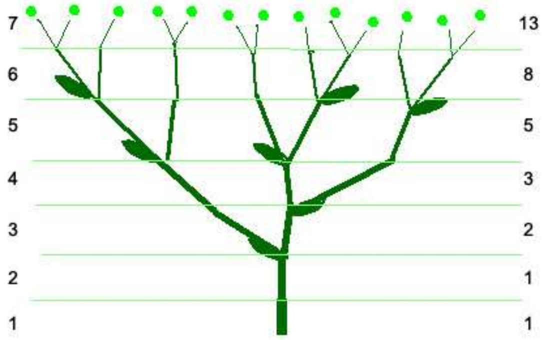
14.1 APPLICATIONS OF HEMACHANDRA-FIBONACCI NUMBERS

The Hemachandra-Fibonacci numbers finds itself in the living world, in biological settings, in art and architecture and photography.

The number of petals found on some flowers follows the Hemachandra-Fibonacci numbers as illustrated below:

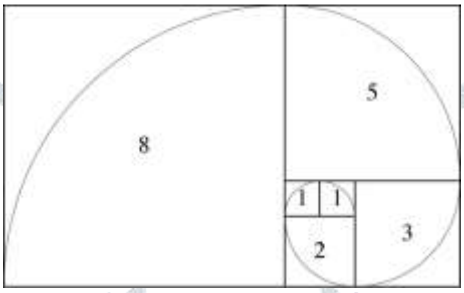


The number of branches of trees always follow the Hemachandra-Fibonacci numbers as illustrated below:



This sequence underpins phyllotaxis (arrangement of leaves on a stem), configuration of inner seeds in a fruit, the family tree of animals and the ancestral code of bees.

One of its most important application is the Fibonacci spiral which approximates the golden spiral. As the figure shows, the sides of the square are subsequent terms of the Hemachandra-Fibonacci numbers.



This spiral exists in nature as illustrated below:



the bracts of pine cone sunflower heads seed arrangement satellite images of hurricanes



Spiral galaxies Tendrils of a stem Nautilus shell

14.1.1 GOLDEN RATIO

As the number of terms in the Hemachandra-Fibonacci sequence tends to infinity, the ratio between the subsequent terms tends to the golden ratio (1.618033…) i.e.

1/1 = 1	13/8 = 1.625
2/1 = 2	21/13 = 1.615
3/2 = 1.5	34/21 = 1.619
5/3 = 1.667	55/34 = 1.618
8/5 = 1.6	

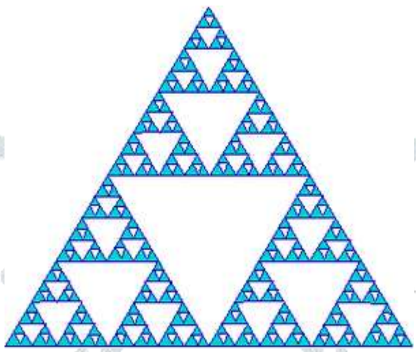
The golden ratio is considered as a “universal law” and was known as “The Divine Proportion” during the Renaissance period. This ratio was found to be used to achieve balance and beauty in many of the famous paintings such as the Mona Lisa, The Last Supper and the Vitruvian Man.

The golden ratio exists in famous forms of architecture such as in The Great Pyramid of Giza in Egypt, Parthenon, a former temple in Greece and in the Taj Mahal in India.

The ratio also exists in the structure of the human chromosome, in the growth of an embryo and even in micromolecular structures such as the revolution of electrons around the nucleus.
The golden ratio and Fibonacci spiral assists in creating attractive photographs that have a strong composition and are pleasing to the eye.
Musical frequencies are also based on this ratio and the sequence was found to exist in the notes of famous musicians such as Mozart and Beethoven. The sequence in combination with phi is used in the design of violins and high-quality speakers.

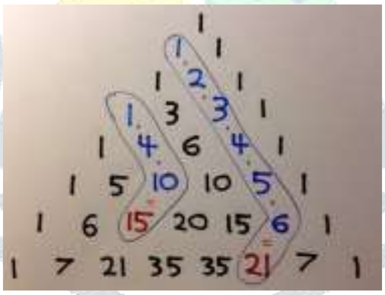
15. SIERPINSKI TRIANGLE

The Sierpinski triangle is a fractal with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. It is also called the Sierpinski gasket or the Sierpinski sieve. It is named after the Polish mathematician Waclaw Sierpinski who described this in 1915, even though it appeared as a decorative pattern as early as in the 13th century.
This triangle can be obtained from the Pascal's triangle with 2^n rows by colouring the even numbers white and the odd numbers blue as illustrated below.



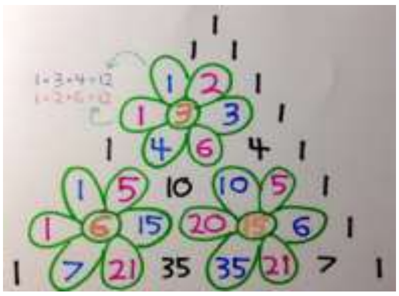
16. HOCKEY STICK ADDITION

A hockey stick pattern exists in the Pascal's triangle which consists of a diagonal string of numbers and a terminating offset number. The offset number equals the sum of the numbers in the diagonal string, and hence it is called hockey stick addition. The hockey stick pattern should begin with one and is found to exist everywhere in the Pascal's triangle.
Two examples are illustrated below:



17. PASCAL'S FLOWER

Six entries or cells join together to form the petals of the Pascal's flower such that the prime factor and the product of the numbers in the alternate petals are equal.



For example, consider the first flower

1*3*4=12(product of numbers coloured blue)

1*2*6=12(product of numbers coloured pink)

The common factor of 1,3,4 and that of 1,2,6 equals 1.

III. LOST IN PASCAL'S TRIANGLE

'Lost in Pascal's triangle' developed by Super Nature Design, a Shanghai based multidisciplinary design company consists of 100 triangular LED lights which are supported within a framework of layered fluorescent triangles. It is an interactive lighting piece which enable individuals to generate series of music and lighting sequences.

IV. CONCLUSION

Thus, the Pascal's triangle exhibits various mathematical concepts and number patterns. It is a simple, yet complex structure which incorporates various spheres of the mathematics and finds applications in combinatorics and probability. Such upright number patterns accord striking signals in making the study of mathematics more interesting and comprehensible.

V. REFERENCES

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