SOFTLY-REGULAR SPACES IN TOPOLOGICAL SPACES

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Abstract: The aim of the present paper is to introduce two new classes of separation axioms (namely partly-regular and softly-regular spaces) in topological spaces which are weaker than regularity and lie between regularity and weakly-regularity. The relationships among strongly rg-regularity, regularity, soft-regularity, partly-regularity, almost regularity and weakly regularity are investigated. Some properties of softly regular spaces in the forms of subspace, product spaces and quotient spaces are obtained. Moreover, we obtained some characterizations of softly regular spaces with π–normality and quasi-normality.

1. Introduction

In 1937, Stone [**11**] introduced the notion of semi-regular spaces and obtained their characterizations. In 1958, Kuratowski [**4**] introduced a generalization of closed sets, called regularly-open and regularly-closed sets in general topology. In 1968, Zaitsev [**14**] introduced the concepts of π-open and π-closed sets and utilizing these sets, introduced the notion of quasi normal spaces and obtained their characterizations and preservation theorems. In 1969, Singal and Arya [**7**] introduced a new class of separation axiom (named almost-regular space) in topological spaces which is weaker than regularity but it is equivalent to semi-regular spaces due to Stone [**11**]. In 1973, Singal and Singal [**10**] introduced the notion of mildly normal spaces and obtained their characterizations. In 2008, Kalantan [**3**] introduced the notion of π-normal spaces and obtained their characterizations and preservation theorems. Recently, M. C. Sharma and Hamant Kumar [**6**] introduced the concept of softly-normal spaces and obtained their characterizations.

Key words: π-open and π-closed sets; softly regular, partly-regular, almost-regular, weakly-regular, semi-regular, π-normal, quasi-normal and softly-normal spaces

2010 Mathematics subject classification: 54B05, 54B10, 54B15, 54D10

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by **Cl(A)** and **Int(A)** respectively.

2.1 Definition. A subset A of a topological space (X, ℑ) is said to be **regularly-open** [4] if it is the interior of its own closure or, equivalently, if it is the interior of some closed set or equivalently, A = Int(Cl(A)). A subset A is said to be **regularly-closed** [4] if it is the closure of its own interior or, equivalently, if it is the closure of some open set or equivalently, A = Cl(Int(A)). Clearly, a set is regularly-open iff its complement is regularly-closed. The finite union of regularly open sets is said to be ***π*–open** [14]. The complement of a π-open set is said to be **π-closed** [**14**]. Every regularly open (resp. regularly closed) set is π–open (resp. π-closed).

2.2 Definition. A space X is said to be **paracompact** if every open covering of X admits a locally finite refinement.

2.3 Definition. A space X is said to be **almost-compact** [**1**] if every open covering of X has a finite subcollection the closures of whose members cover X.

2.4 Definition. A space X is said to be **nearly-compact** [**8**] if every open covering of X admits a finite subcollection the interiors of the closures of whose members cover X.

2.5 Definition. A space X is said to be a **Urysohn space** [**13**] if for every pair of distinct points x and y, there exist open sets U and V such that $x \in U$, $y \in V$ and $Cl(U) \cap Cl(V) = \varnothing$.

2.6 Definition. A space X is said to be **π-normal** [**3**] if for every pair of disjoint subsets A, B of X, one of which is closed and the other is π-closed, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

2.7 Definition. A space X is said to be **quasi-normal** [**14**] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

2.8 Definition. A space X is said to be **softly-normal** [**6**] if for every pair of disjoint subsets A, B of X, one of which is π-closed and the other is regularly-closed, there exist disjoint open sets U, V of X such that $A \subset U$ and $B \subset V$.

By the definitions stated above, we have the following diagram [6]:

normality ⇒ π-normality ⇒ quasi-normality ⇒ soft- normality ⇒ mild- normality

Where none of the implications is reversible.

3. Softly regular space

3.1 Definition. A space $(X, \Im)$ is said to be **softly regular** if for every π–closed set A and a point $x \notin A$, there exist disjoint open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \varnothing$.

3.2 Definition. A space $(X, \Im)$ is said to be **partly regular** if for every point x and every π-open set U containing x, there is an open set V such that $x \in V \subset Cl(V) \subset U$.

3.3 Definition. A space $(X, \Im)$ is said to be **almost regular** [7] if for every regularly closed set A and a point $x \notin A$, there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \varnothing$.

3.4 Definition. A space $(X, \Im)$ is said to be **weakly regular [7]** if for every point x and every regularly-open set U containing x, there is an open set V such that $x \in V \subset Cl(V) \subset U$.

3.5 Definition. A space $(X, \Im)$ is said to be **strongly rg-regular** [2] if for every rg-closed set A and a point $x \notin A$, there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \varnothing$.

3.6 Definition. A space $(X, \Im)$ is said to be **semi-regular** [11] if for each point x of the space and each open set U containing x, there is an open set V such that $x \in V \subset Int(Cl(V)) \subset U$.

By the definitions stated above, we have the following diagram:

strongly rg-regularity $\Rightarrow$ regularity $\Rightarrow$ soft-regularity $\Rightarrow$ partly-regularity

$\Downarrow$ $\Downarrow$

almost -regularity $\Rightarrow$ weakly-regularity

Where none of the implications is reversible as can be seen from the following examples:

3.7 Example. Let $X = \{a, b, c\}$ and $\Im = \{\varnothing, \{a\}, \{b, c\}, X\}$. Consider the closed set {b, c} and a point 'a' such that $a \notin \{b, c\}$. Then {b, c} and {a} are disjoint open sets such that $\{b, c\} \subset \{b, c\}$, $a \in \{a\}$ and $\{b, c\} \cap \{a\} = \varnothing$. Similarly, for the closed set {a} and a point 'c' such that $c \notin \{a\}$. Then there exist open sets {a} and {b, c} such that $\{a\} \subset \{a\}$, $c \in \{b, c\}$ and $\{a\} \cap \{b, c\} = \varnothing$. It follows that $(X, \Im)$ is regular as well as softly regular space.

3.8 Example. Let $X = \{a, b, c\}$ and $\Im = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. If we take a point 'a' and an open set V = {a}, then Cl(V) = {a, c} and a regularly-open set U = X. So by the definition of weakly regular space $x \in V \subset Cl(V) \subset U$, where V be an open set and U be a regularly-open set such that $a \in \{a\} \subset \{a, c\} \subset X$. Hence $(X, \Im)$ is weakly regular. If we take a point 'a' and a regularly-closed set A = {b, c} does not containing the point 'a', there donot exist disjoint open sets containing the point 'a' and a regularly-closed set A = {b, c}. Hence $(X, \Im)$ is not almost-regular.

3.9 Example. Let $X = \{a, b, c\}$ and $\Im = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Then $(X, \Im)$ is weakly regular but not partly-regular. If we take a point 'a' and an open set V = {a}, then Cl(V) = {a, c}. Let U = {a, b} be any π–open set. So by the definition of partly-regular space $x \in V \subset Cl(V) \subset U$, where V be an open set and U be a π-open set such that $a \in \{a\} \subset \{a, c\} \not\subset \{a, b\}$. Hence $(X, \Im)$ is not partly-regular.

3.10 Example. Let $X = \{a, b, c\}$ and $\Im = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Then $(X, \Im)$ is weakly regular but not softly-regular. Let A = {c} be any π–closed set doesnot containing a point 'a' i.e. $a \notin \{c\}$, there do not exist disjoint open sets containing the point 'a' and the π–closed set A = {c}. Hence $(X, \Im)$ is not softly-regular.

3.11 Example. Let $X = \{a, b, c, d\}$ and $\Im = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then the space $(X, \Im)$ is almost regular but not strongly rg-regular. If we take a point 'a' and F = {b} be any rg-closed set. Then there do not exist disjoint open sets containing the point 'a' and rg–closed set F = {b}. Hence $(X, \Im)$ is not strongly rg-regular.

3.12 Theorem. A space is partly–regular X iff for every point x and every π–open set U containing Cl({x}), there is an open set V, $x \in V \subset Cl(V) \subset U$.

3.13 Theorem. For a topological space $(X, \Im)$, the following properties are equivalent:
(a) (X, τ) is softly-regular.

(b) For every $x \in X$ and every π–open set U containing x, there exists an open set V such that $x \in V \subset Cl(V) \subset U$.
(c) For every π–closed set A, the intersection of all the closed neighbourhood of A is A.
(d) For every set A and a π–open set B such that $A \cap B \neq \varnothing$, there exists an open set F such that $A \cap F \neq \varnothing$ and $Cl(F) \subset B$.
(e) For every nonempty set A and π–closed set B such that $A \cap B = \varnothing$, there exists disjoint open sets L and M such that $A \cap L \neq \varnothing$ and $B \subset M$.

Proof.
(a) $\Rightarrow$ (b). Suppose (X, τ) is softly-regular. Let $x \in X$ and U be a π–open set containing x so that $X – U$ is π–closed. Since $(X, \Im)$ is softly regular, there exist open sets V_1 and V_2 such that $V_1 \cap V_2 = \varnothing$ and $x \in V_1$, $X – U \subset V_2$. Take $V = V_1$. Since $V_1 \cap V_2 = \varnothing$, $V \subset X – V_2 \subset U$ that implies $Cl(V) \subset Cl(X- V_2) = X – V_2 \subset U$. Therefore $x \in V \subset Cl(V) \subset U$.

(b) $\Rightarrow$ (c). Let A be π-closed set and $x \notin A$. Since A is π-closed, X - A is π-open and $x \in X – A$. Therefore by (b) there exists an open set V such that $x \in V \subset Cl(V) \subset X – A$. Thus $A \subset X - Cl(V) \subset X – V$ and $x \notin X – V$. Consequently X - V is a closed neighborhood of A

(c) $\Rightarrow$ (d). Let $A \cap B \neq \varnothing$ and B be π-open. Let $x \in A \cap B$. Since B is π-open, X - B is π-closed and $x \notin X – B$. By our assumption, there exists a closed neighborhood V of X - B such that $x \notin V$. Let $X - B \subset U \subset V$, where U is open. Then $F = X - V$ is open such that $x \in F$ and $A \cap F \neq \varnothing$. Also X - U is closed and $Cl(F) = Cl(X-V) \subset X - U \subset B$. This shows that $Cl(F) \subset B$.

(d) $\Rightarrow$ (e). Suppose $A \cap B = \varnothing$, where A is non-empty and B is π-closed. Then X - B is π-open and $A \cap (X – B) \neq \varnothing$. By (d), there exists an open set L such that $A \cap L \neq \varnothing$, and $L \subset Cl(L) \subset X - B$. Put $M = X - Cl(L)$. Then $B \subset M$ and L, M are open sets such that $M = X - Cl(L) \subset (X - L)$.

(e) $\Rightarrow$ (a). Let B be π-closed and $x \notin B$. Then $B \cap \{x\} = \varnothing$. By (e), there exist disjoint open sets L and M such that $L \cap \{x\} \neq \varnothing$ and $B \subset M$. Since $L \cap \{x\} \neq \varnothing$, $x \in L$. This proves that $(X, \Im)$ is softly-regular.

3.14 Theorem. A topological space $(X, \Im)$ is softly-regular if and only if for each π-closed set F of $(X, \Im)$ and each $x \in X – F$, there exist open sets U and V of $(X, \Im)$ such that $x \in U$ and $F \subset V$ and $Cl(U) \cap Cl(V) = \varnothing$.

Proof: Let F be a π-closed set in $(X, \Im)$ and $x \notin F$. Then there exist open sets U_x and V such that $x \in U_x$, $F \subset V$ and $U_x \cap V = \varnothing$. This Implies that $U_x \cap Cl(V) = \varnothing$. Since Cl(V) is closed and $x \notin Cl(V)$. Since $(X, \Im)$ is softly-regular, there exist open sets G and H of $(X, \Im)$ such that $x \in G$, $Cl(V) \subset H$ and $G \cap H = \varnothing$. This implies $Cl(G) \cap H = \varnothing$. Take $U = U_x \cap G$. Then U and V are open sets of $(X, \Im)$ such that $x \in U$ and $F \subset V$ and $Cl(U) \cap Cl(V) = \varnothing$, since $Cl(U) \cap Cl(V) \subset Cl(G) \cap H = \varnothing$. Conversely, suppose for each π-closed set F of $(X, \Im)$ and each $x \in X – F$, there exist open sets U and V of $(X, \Im)$ such that $x \in U$, $F \subset V$ and and $Cl(U) \cap Cl(V) = \varnothing$. Now $U \cap V \subset Cl(U) \cap Cl(V) = \varnothing$. Therefore $U \cap V = \varnothing$. Thus $(X, \Im)$ is softly-regular.

4. Softly-regularity with some other separation axioms

4.1 Theorem [7]. Every almost-regular, semi-regular space is regular.

4.2 Corollary. Every softly-regular, semi-regular space is regular.
Proof. Using the fact that every softly-regular space is almost-regular.

4.3 Theorem [7]. Every almost-regular, Hausdorff space is a Urysohn space.

4.4 Corollary. Every softly-regular, Hausdorff space is a Urysohn space.
Proof. Using the fact that every softly-regular space is almost-regular.

4.5 Remark. Every regular T_0-space is T_1. However, the situation is different in this respect for softly-regular spaces. A softly-regular T_0-space need not be T_1 as the following example shows.

4.6 Example. Let X = {a, b, c} and let = {∅, {a}, {a, b}, {a, c}, X}. Then (X, $\Im$) is softly-regular T_0-space which is not T_1.

4.7 Theorem. Every partly-regular, paracompact space is π-normal.
Proof. Let X is a partly-regular, paracompact space and let A be a closed subset of X. Let B be a π-closed set such that A ∩ B = ∅. Now, for each point x ∈ A, Cl{x} is contained in A. Therefore, X – B is a π–closed set containing Cl{x}. Since X is partly-regular, there is an open set V_x containing x such that $Cl(V_x) \cap B = \emptyset$. The family of subsets $\{V_x : x \in A\} \cup \{X – A\}$ is an open covering of X and so there is a locally finite open refinement of it. Suppose that $\eta = \{U_\alpha\}$ denotes the members of that family which have nonempty intersections with A. Let $U_1 = \cup_\alpha U_\alpha$, which is clearly an open set containing A. Let $U_2 = X - \cup_\alpha Cl(U_\alpha)$ which is an open set, because $\{U_\alpha\}$ being locally finite, $\cup_\alpha Cl(U_\alpha) = Cl(\cup_\alpha U_\alpha)$. Also, $U_1 \cap U_2 = \emptyset$. Since η is a refinement and each member of it intersects A, for each U_α there must exist an x ∈ A such that $U_\alpha \subset V_x$. Now, since $Cl(U_\alpha) \subset Cl(V_x) \subset X – B$, $B \subset X – V_x \subset X - U_\alpha$ for every $U_\alpha \in \eta$. Hence, $B \subset U_2$ and so X is π–normal.

Using the fact that every compact space is paracompact and every softly-regular space is partly-regular, we have following corollaries:

4.8 Corollary. Every softly-regular, paracompact space is π-normal.

4.9 Corollary. Every partly-regular, compact space is π-normal.

4.10 Corollary. Every softly-regular, compact space is π-normal.

Using the fact that a regular closed subset of an almost compact space is almost compact (in the relativised topology), Single and singal [**10**] proved that an almost regular almost compact space is mildly normal. Lal and Rahman [**5**] also proved that if X is a almost regular space in which every π–closed set is almost compact, then X is quasi normal. We improve upon this result to read

4.11 Theorem. If X is a softly regular space in which every π–closed set is almost compact, then X is quasi normal.

Proof. Let A and B be two disjoint π-closed sets, and let $x \in A$. Then $x \notin B = \bigcap_{i=1}^{n} B_i$, where B_i are regularly closed sets. Thus $x \notin B_j$ for some regularly closed set B_j. Since X is softly-regular, there exist open sets G_x and H_x such that $x \in G_x$, $B \subset B_j \subset H_x$ and $Cl(G_x) \cap Cl(H_x) = \varnothing$. Now $\{G_x \cap A : x \in A\}$ is a cover of A by sets open in A. Since A is almost compact, there exist a finite subcollection $\{G_{xi} \cap A : i = 1, 2, 3, \ldots, m\}$ closures of whose members cover A. But in this case $A \subset \cup_{i=1}^{m} Cl(G_{xi})$. Also if $H = \cap_{i=1}^{m} H_{xi}$, and $G = X - \cap_{i=1}^{m} Cl(H_{xi})$, then G and H are disjoint open set enclosing A and B respectively.

Lal and Rahman [**5**] observed that if A is a π-closed set and $x \notin A$, then there exists a regularly closed set B such that $A \subset B$ and $x \notin B$. We thus have following useful characterization of softly-regularity.

4.12 Theorem. X is a softly-regular if and only if every π–closed set A and every point $x \notin A$ are separated by open sets.
Proof. Easy to verify.

4.13 Theorem. Every softly-regular, Lindelof space is quasi-normal.
Proof. Let X be a softly-regular, Lindelof space and let A and B be two disjoint π–closed subsets of X. For each $x \in A$, there exists an open set U_x such that $x \in U_x \subset Cl(U_x) \subset X - B$. It follows that for each point $x \in A$, there is an open set U_x such that $x \in U_x$ and $Cl(U_x) \cap B = \varnothing$. Then $\eta = \{U_x : x \in A\}$ is an open covering of A. Since every closed subset of a Lindelof space is Lindelof, therefore η admits of a countable subcovering $\{U_n : n = 1, 2, \ldots\}$. Similarly, for each point $y \in B$, there exists an open set V_y such that $y \in V_y$ and $Cl(V_y) \cap A = \varnothing$. Again $\upsilon = \{ V_y : y \in B\}$ is an open covering of the Lindelof set B and therefore υ has a countable subcovering $\{V_n : n = 1, 2, \ldots\}$. Let $A_n = U_n - \cup \{Cl(V_k) : k \leq n\}$and let $B_n = V_n - \cup\{Cl(U_k) : k \leq n\}$for each $n = 1, 2, \ldots$. Since $A_n \cap V_m = \varnothing$ for all $n \geq m$, therefore $A_n \cap B_m = \varnothing$ for all $n \geq m$. Similarly, $A_n \cap B_m = \varnothing$ for all $n \leq m$. Hence $A_n \cap B_m = \varnothing$ for all m, n. If $G = \cup \{A_n : n = 1, 2, \ldots\}$ and $H = \cup \{B_n : n = 1, 2, \ldots\}$, then G and H are disjoint open sets such that $A \subset G$, $B \subset H$. Hence X is quasi-normal.

4.14 Theorem. A softly-regular space with a σ–locally finite base is quasi-normal.
Proof. Easy to verify.

4.15 Theorem. A softly-regular nearly paracompact space is quasi-normal.
Proof. Easy to verify.

5. Soft-regularity in subspace, product spaces and quotient spaces

This is known that regularity is a hereditary property. But unfortunately, this is not the case for soft-regularity. In fact, soft-regularity is not even weakly hereditary, that is, even a closed subspace of a softly-regular space may fail to be softly-regular. We can see it by following example.

5.1 Example. Let X = {a, b, c, d} and let $\Im = \{\varnothing, \{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}, X\}$. Then (X,) is almost-regular, since $\varnothing$ and X are the only π-open sets. Now, consider the closed subspace Y = {a, b, c} and the subspace $\Im_Y = \{\varnothing, \{b\}, \{b, c\}, X\}$. Let A = {a, c} be any π-closed set in Y doesnot contain the point b i.e b

$\notin$ {a, c}. Then there exist no disjoint open subsets of Y which contain the point b and π–closed set A respectively and therefore Y is not softly-regular.

5.2 Theorem. Every regularly-open subspace of a softly-regular space is softly-regular.
Proof. Easy to verify.

5.3 Corollary. Every subspace of a softly-regular space which is both open and closed is softly-regular.
Proof. Any subspace which is both open and closed is regularly-open and hence the result follows from **Theorem 5.2**.

5.4 Corollary. Every regularly-closed subspace of a softly-regular, extremally-disconnected space is softly-regular.
Proof. Since in an extremally-disconnected space, closure of an open set is open, therefore in such a space every regularly-closed set is regularly-open and hence the result follows from **Theorem 5.2**.

In the following theorem, we study the behavior of soft-regularity with regard to products. We know that the product of an arbitrary family of topological spaces is regular iff each factor is regular. The same is true for softly-regular spaces.

5.5 Theorem. The product of an arbitrary family of topological spaces is softly-regular iff each factor space is softly-regular.
Proof. Use the same construction as in the proof of the result for regular space.

5.6 Theorem. Let (X, $\Im$) be any topological space and define R by setting xRy iff Cl{x} = Cl{y}. Then R is an equivalence relation in X. If $\upsilon : X \rightarrow X/R$ be the projection map of X onto the quotient space X / R, then X / R is softly-regular if X is so.
Proof. Use the same construction as in the proof of the result for regular space.

Conclusion. In this paper, we introduced two new classes of separation axioms (namely partly-regular and softly-regular spaces) in topological spaces which are weaker than regularity and lie between regularity and weakly-regularity. The relationships among strongly rg-regularity, regularity, soft-regularity, partly-regularity, almost regularity and weakly regularity are investigated. Some properties of softly regular spaces in the forms of subspace, product space and quotient space are obtained. We also obtained some characterizations of softly regular spaces with π–normality and quasi-normality. We can obtain more characterizations of softly regular spaces with soft–normality and mild-normality.

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