BEST PROXIMITY OF A MAP SATISFYING GEOMETRIC MEAN CONDITION

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Abstract. Let A and B be any two nonempty weakly compact convex subsets of a Banach space X. In this paper, a new class of cyclic map, $T : A \cup B \to A \cup B$ satisfying geometric mean condition, is introduced and used to investigate the existence of a point $x \in A$, such that $d(x, Tx) = d(A, B)$, known as best proximity points. If $A = B$ then our result proves the existence of fixed point proved by Khan.

Keywords: Best proximity points, geometric mean condition, cyclic map, proximal normal structure, strictly convex.
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1. Introduction

In [2], Khan introduced a map $T : X \to X$ satisfying the condition $d(Tx, Ty) \leq \{d(x, Tx)d(y, Ty)\}^{\frac{1}{2}}$ and proved that in a reflexive Banach space X the map $T : K \to K$ has a unique fixed point, where K is a nonempty, bounded, closed and convex set having normal structure. The author also proved that if X is a strictly convex reflexive Banach space and K a bounded closed and convex subset of X then $T : K \to K$ has a unique fixed point. Once this is established then it is easy to observe that the self map T must be a constant map. Hence this map T has a fixed point if and only if T is constant. Suppose T is non-constant, then for any $x \in K$, $d(x, Tx) > 0$. In such a situation, it will be interested to search for a point x such that $d(x, Tx)$ is minimum in some sense. In this paper, we consider two nonempty weakly compact convex subsets A, B of a Banach space X and a cyclic map $T : A \cup B \to A \cup B$ ie., $T(A) \subseteq B$ and $T(B) \subseteq A$, satisfying the above condition and prove that there exists a point $x_0 \in A \cup B$ such that $d(x_0, Tx_0)$ is minimum. That is, $d(x_0, Tx_0) = \inf\{\|x - y\| : x \in A, y \in B\} = d(A, B)$. We call such points $x_0 \in A \cup B$ as best proximity point of T.

2. Preliminaries

Let us first recall some definitions and notations, used in this paper. Let A, B be any two subsets of a normed linear space X, then

$$\delta(A, B) = \sup\{\|x - y\| : x \in A, y \in B\};$$

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$$d(A, B) = \inf\{\|x - y\| : x \in A, y \in B\};$$

$$A_0 = \{x \in A: \|x - y'\| = d(A,B) \text{ for some } y' \in B\};$$

$$B_0 = \{y \in B: \|x - y'\| = d(A,B) \text{ for some } x' \in A\}.$$

A pair (A, B) of subsets of a normed linear space is said to be a proximal pair if for each $(x, y) \in A \times B$

There exists $(x',y') \in A \times B$ such that $\|x - y'\| = \|x' - y\| = d(A,B)$. In[1], Eldredet.al., introduced the following notion called proximal normal structure. In the definition, we say that a pair (A,B) satisfies a property if each of the sets A and B has that property. A convex pair (K_1, K_2) in a Banach space is said to have proximal normal structure if for any closed, bounded, convex proximal pair $(H_1,H_2) \subset (K_1,K_2)$ for which $d(H_1, H_2) = d(K_1, K_2)$ and $\delta(H_1, H_2) > d(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that $\delta(x_1, H_2) < \delta(H_1, H_2)$, $\delta(x_2, H_1) < \delta(H_1, H_2)$. Note that the pair (K, K) has proximal normal structure if and only if K has normal structure in the sense of Brodskii and Milman[4].

In [3], Kirk et.al., proved that, if the pair (A, B) is nonempty, weakly compact and convex then the pair (A_0, B_0) also has the same properties and moreover, $d(A_0, B_0) = d(A, B)$. Also, a Banach space X is said to be strictly convex if for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ then $\left\|\frac{x+y}{2}\right\| < 1$. In [5], SankarRaj et.al. proved that a normed linear space X is strictly convex if and only if X has the P -property. A pair (A, B) of nonempty subsets of a normed linear space X is said to have P -property if and only if $\|x_1 - y_1\| = d(A, B)$ and $\|x_2 - y_2\| = d(A, B)$ implies $\|x_1 - x_2\| = \|y_1 - y_2\|$ whenever $x_1, x_2 \in A$ and $y_1, y_2 \in B$. A normed linear space X is said to have the P -property if and only if every pair (A, B) of nonempty and closed convex subsets of X has the P -property. Let us now define a new class of cyclic map as follows.

Deftnition 2.1. A cyclic map $T : A \cup B \to A \cup B$ is said to be a map satisfying geometric mean condition if $\|Tx - Ty\| \leq \{\|x - Tx\| \, \|y - Ty\|\}^{\frac{1}{2}}$ for all $x \in A$ and $y \in B$.

Before proving our main result, let us prove the following lemma.

Lemma 2.2. *Let T be a cyclic map satisfying the geometric mean condition. Then for each* $x \in A \cup B$ *A nd for any positive integer n,* $\|T^n x - T^{n+1}x\| \leq \|T^{n-1}x - T^n x\|$.

Proof. Let n be any positive integer and $x \in A \cup B$. Suppose $\|T^n x - T^{n+1}x\| = 0$ then the proof follows immediately.

Otherwise $\|T^n x - T^{n+1}x\| = \|T(T^{n-1}x) - T(T^n x)\| \leq \{\|T^{n-1}x - T^n x\| \, \|T^n x - T^{n+1}x\|\}^{\frac{1}{2}}$. Squaring both sides we get $\|T^n x - T^{n+1}x\|^2 \leq \|T^{n-1}x - T^n x\| \, \|T^n x - T^{n+1}x\|$ which implies $\|T^n x - T^{n+1}x\| \leq \|T^{n-1}x - T^n x\|$.

Note that if we take $n=1$ in the above lemma, then the inequality becomes $\|Tx - T^2x\| \leq \|x - Tx\|$.

3. MainResult

Theorem 3.1. *Let A and B be a pair of nonempty weakly compact convex subsets of a Banach space X and suppose* (A, B) *has proximal normal structure. Let T be the map defined as in definition 2.1 and* $T(A_0) \subseteq B_0$ *and* $T(B_0) \subseteq A_0$. *Then T has a best proximity point.*

Proof. Let A_0 and B_0 be the proximal pair associated with A and B. Hence A_0 and B_0 are weakly compact and convex. Let $\Gamma = \{F \subseteq A_0 \cup B_0: F \cap A_0$ and $F \cap B_0$ are nonempty closed and convex $T(F \cap A_0) \subseteq F \cap B_0$, $T(F \cap B_0) \subseteq F \cap A_0$ and $d(F \cap A_0, F \cap B_0) = d(A, B)\}$. Since $A_0 \cup B_0 \in \Gamma$, Γ is nonempty. Let $\{F_\alpha\}_{\alpha \in J}$ be a descending chain in Γ, and let $F_0 = \cap_\alpha F_\alpha$. Then $F_0 \cap A_0 = \cap_\alpha (F_\alpha \cap A_0)$. So $F_0 \cap A_0$ is nonempty closed and convex. Similarly, $F_0 \cap B_0$ is nonempty closed and convex. Also $T(F_0 \cap A_0) \subseteq F_0 \cap B_0$, $T(F_0 \cap B_0) \subseteq F_0 \cap A_0$. Now to show that $F_0 \in \Gamma$ we need to show that $d(F_0 \cap A_0, F_0 \cap B_0) = d(A, B)$. However for each $\alpha \in J$, it is possible to select $x_\alpha \in F_\alpha \cap A_0$ and

$y_\alpha \in F_\alpha \cap B_0$ such that $\|x_\alpha - y_\alpha\| = d(A,B)$. It is also possible to choose weakly convergent subnets $\{x_{\alpha'}\}$ and $\{y_{\alpha'}\}$ (with the same indices); say weak-limit $x_{\alpha'} = x$ and weak-lim $y_{\alpha'} = y$. Then clearly $x \in F_0 \cap A_0$ and $y \in F_0 \cap B_0$. By weak lower semicontinuity of the norm, $\|x - y\| \le d(A, B)$.

Now, $d(A,B) \le d(F_0 \cap A_0, F_0 \cap B_0) \le \|x-y\| \le d(A,B)$. Since every chain in Γ is bounded below by a member of Γ, Zorns lemma implies that Γ has a minimal element, say K. Let $K_1 = K \cap A_0$ and $K_2 = K \cap B_0$. If $\delta(K_1, K_2) = d(K_1, K_2)$, then $\|x - Tx\| = d(K_1, K_2) = d(A, B)$ for any $x \in K_1$, and we are finished. Suppose $\delta(K_1,K_2) > d(K_1,K_2)$, we complete the proof by showing that this leads to a contradiction. By proximal normal structure, there exists $(y_1,y_2) \in K_1 \times K_2$ and $\beta \in (0,1)$ such that $\delta(y_1,K_2) \le \beta\delta(K_1,K_2)$ and $\delta(y_2,K_1) \le \beta\delta(K_1,K_2)$. Since (K_1,K_2) is a proximal pair, there exists $(y'_1, y'_2) \in K_1 \times K_2$ such that $\|y_1 - y'_2\| = \|y_2 - y'_1\|^2 = d(K_1,K_2)$.[1] So for any $z \in K_2$, $\left\|\frac{y_1+y_1'}{2} - z\right\| \le \left\|\frac{y_1-z}{2}\right\| + \left\|\frac{y_1'-z}{2}\right\| \le \beta\frac{\delta(K_1,K_2)}{2} + \frac{\delta(K_1,K_2)}{2} = \alpha\frac{\delta(K_1,K_2)}{2}$ where $\alpha = \frac{1+\beta}{2} \in (0,1)$. Let $x_1 = \frac{y_1+y_1'}{2}$ and $x_2 = \frac{y_2+y_2'}{2}$.

Then $\delta(x_1,K_2) \le \alpha\delta(K_1,K_2)$ and $\delta(x_2,K_1) \le \alpha\delta(K_1,K_2)$ and $\|x_1 - x_2\| = d(K_1,K_2)$. Hence there exists $r > 0$ such that $\delta(x_1, K_2) \le r < \delta(K_1, K_2)$ and $\delta(x_2, K_1) \le r < \delta(K_1, K_2)$. Define $L_1 = \{x \in K_1 : \|x - Tx\| \le r\}$; and $L_2 = \{y \in K_2 : \|y - Ty\| \le r\}$. Since K_1 is convex, $x_1 \in K_1$ and $\delta(x_1,K_2) \le r$ implies $\|x_1 - Tx_1\| \le r$. Hence $x_1 \in L_1$ similarly we can show that $x_2 \in L_2$. Thus L_1 and L_2 are nonempty sets and $d(L_1, L_2) = d(A, B)$. Let $P_1 = \overline{Co(T(L_1))}$ and $P_2 = = \overline{Co(T(L_2))}$. Clearly P_1 and P_2 are nonempty closed and convex and $d(P_1, P_2) = d(A, B)$. Let us now show that $T(P_1) \subseteq P_2$ and $T(P_2) \subseteq P_1$. Let us first prove that $T(P_1) \subseteq P_2$ by splitting it into three different cases. Choose $y \in P_1$

Case(i) Let $y = T(p)$ where $p \in L_1$. Clearly $T(p) \in K_2$ and $\|Tp - T^2p\| \le \|p - Tp\| \le r$. Hence $Tp \in L_2$ which implies $Ty = T^2p \in T(L_2)$. That is $Ty \in P_2$.

Case(ii) Let $y = \sum_{i=1}^{n} \lambda_i T p_i$ where $p_i \in L_1$. Clearly $y \in K_2$ and $\|y - Ty\| \le \sum_{i=1}^{n} \lambda_i \|Tp_i - Ty\| \le \sum_{i=1}^{n} \lambda_i \{\|p_i - Tp_i\|\|y - Ty\|\}^{\frac{1}{2}} \le \sum_{i=1}^{n} \lambda_i \{r\|y - Ty\|\}^{\frac{1}{2}}$ which implies $\|y - Ty\| \le r$. Hence $y \in L_2$ and $Ty \in P_2$.

Case (iii) Let $y \in \overline{Co(T(L1))}$ then there exists a sequence y_k in $Co(T(L_1))$ of the form $y_k = \sum_{i=1}^{n_k} \lambda_i T p_i$. For any $\epsilon > 0$ there exists a positive integer k such that $\|y_k - y\| < \epsilon$. Now $\|Ty - y\| \le \|Ty - y_k\| + \|y_k - y\| \le \left\|Ty - \sum_{i=1}^{n_k} \lambda_i T p_i\right\| + \epsilon$. Since ϵ is arbitrary, we get $\|Ty - y\| \le \left\|Ty - \sum_{i=1}^{n_k} \lambda_i T p_i\right\| \le \sum_{i=1}^{n_k} \lambda_i \|Ty - Tp_i\| \le \sum_{i=1}^{n_k} \lambda_i \{\|p_i - Tp_i\|\|y - Ty\|\}^{\frac{1}{2}} \le \sum_{i=1}^{n_k} \lambda_i \{r\|y - Ty\|\}^{\frac{1}{2}}$ which implies $\|y - Ty\| \le r$. Hence $y \in L_2$ and $Ty \in P_2$. Thus we have shown that $T(P_1) \subseteq P_2$ similarly we can show that $T(P_2) \subseteq P_1$. Thus $P = P_1 \cup P_2 \in \Gamma$ by the minimality of K, $\delta(K_1, K_2) \le \delta(P_1, P_2)$. But $\delta(P_1, P_2) = \delta(\overline{Co(T(L1))}, \overline{Co(T(L2))}) = \delta(Co(TL_1), Co(TL_2)) = \delta(TL_1, TL_2) = \sup\{\|Tx - Ty\| : x \in L_1, y \in L_2\} \le \sup\{\{\|x - Tx\|\|y - Ty\|\}^{\frac{1}{2}} : x \in L_1, y \in L_2\} \le r$. Which is a contradiction to the fact that $r < \delta(K_1, K_2)$.

Thus we have $\delta(K_1, K_2) = d(K_1,K_2)$. (3.1)

If we take $A = B$ in Theorem 3.1 then by (3.1), $\delta(K_1, K_2) = d(K_1, K_2) = 0$. Thus we have the following theorem.

Corollary 3.2. [2, Theorem 8] *Let A be a nonempty weakly compact convex subset of a Banach space X and let A has normal structure. If T is a mapping of A into itself such that* $\|Tx - Ty\| \le \{\|x - Tx\|\|y - Ty\|\}^{\frac{1}{2}}$ *for all x, y ∈ X, then T has a unique fixed point.*

Note that in Theorem 3.1, the best proximity point of T is not necessarily unique. Let us now see an example to illustrate this.

Example 3.3. *Let* $X = (\mathrm{R}^2, \|.\|_\infty)$, *where* $\|(x,y)\|_\infty = \max\{|x|, |y|\}$ *for each* $(x,y) \in \mathrm{R}^2$ *and* $A = \{(-1, y) : 0 \leq y \leq 1\}$, $B = \{(1, y) : 0 \leq y \leq 1\}$. *Then every cyclic map* $T : A \cup B \rightarrow A \cup B$ *will satisfy 2.1 and each point of* $A \cup B$ *is a best proximity point.*

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The following theorem discusses the case in which the best proximity is unique.

Theorem 3.4. *Let A and B be a pair of nonempty weakly compact convex subset of a strictly convex Banach space X. Let T be the map defined as in definition 2.1 and* $T(A_0) \subseteq B_0$ *and* $T(B_0) \subseteq A_0$. *Then T has a unique best proximitypoint.*

Proof. Construct K_1 and K_2 as in Theorem3.1. If one of the set K_1 or K_2 is singletonor if $\delta(K_1,K_2) = d(K_1, K_2)$, then $\|x - Tx\| = d(K_1, K_2) = d(A, B)$ for any $x \in K_1$, and we are finished. Let us now consider the case such that both K_1 and K_2 are not singleton and $\delta(K_1, K_2) > d(K_1, K_2)$. Let x' and y' be any two distinct points in K_1 then correspondingly there exists two distinct points a' and b' in K_2 such that $\|x' - a'\| = \|y' - b'\| = d(A,B)$. Clearly $\left\|\frac{x'+y'}{2} - \frac{a'+b'}{2}\right\| = d(A,B)$. Also by strictly convexity $\left\|x' - T\left(\frac{x'+y'}{2}\right)\right\| \leq \delta(A,B)$ and $\left\|y' - T\left(\frac{x'+y'}{2}\right)\right\| \leq \delta(A,B)$ implies $\left\|\frac{x'+y'}{2} - T\left(\frac{x'+y'}{2}\right)\right\| < \delta(A,B)$. Let $z = \frac{x'+y'}{2}$, clearly $z \in K_1$ and $\|z - Tz\| \leq r_1 < \delta(A,B)$. Similarly we can find an element $z_2 \in K_2$ such that $\|z_2 - Tz_2\| \leq r_2 < \delta(A,B)$. Let $r = \max\{r_1, r_2\}$. Define $L_1 = \{x \in K_1 : \|x - Tx\| \leq r\}$ and $L_2 = \{y \in K_2 : \|y - Ty\| \leq r\}$. Then proceed as in Theorem 3.1. Let us now show that the best proximity point is unique. Suppose there exists two best proximity points say x, y then $\|x - Tx\| = \|y - Ty\| = d(A,B)$ by Lemma2.2, $\|T^2y - Ty\| = d(A,B)$. Hence by P-property $y = T^2y$. Also $\|T^2y - Tx\| \leq \{\|Ty - T^2y\| \|x - Tx\|\}^{\frac{1}{2}}\}$. Hence $\|T^2y - Tx\| = d(A,B)$. Again by P-property, we get $x = T^2y$ and hence $x = y$.

If we take $A = B$ in Theorem 3.4, then we get the following corollary.

Corollary 3.5. [2, Theorem 9] *Let A be a nonempty weakly compact and convex subset of a strictly convex Banach space X. Let T be a mapping of A into itself such that* $\|Tx - Ty\| \leq \{\|x - Tx\| \|y - Ty\|\}^{\frac{1}{2}}$ *for all* $x, y \in X$, *then T has a unique fixed point.*

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