SPECTRAL THEORY IN HILBERT SPACE

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ABSTRACT:

We now proceed to give a more abstract from of the spectral theorem ,based on the representation theorem for commutative B*-algebras.A spectral measure on a Hilbert space H is a mapping and from the σ-algebra of boral measurable subsets of C into L(H).there are types of Hilbert space the finite dimensional.A normed algebra is a normed vector space and an algebra.The spectral theorem of the continuous functional calculus to normal operators.

KEYWORDS:

Spectral, complex, product, operators, banach space, bounded operators, function, linear algebra, continuous , normal operators.

INTRODUCTION:

The spectral theory of operators on a Hilbert space.The diagonalization of asymmetric metric may be given several interpretations.Our approach to the spectral theorem will be by way of the study of c*-algebras.A spectral measure on a Hilbert space H is a mapping and from the σ-algebra of boral measurable subsets of C into L(H).In this notes we provide an introduction to compact linear operators on banach and Hilbert spaces.The construction of the measure is particularly simple when the algebra of operators has a so called cyclic vector.

There are really three types of Hilbert spaces the finite dimensional ones, essentially just ځn and two infinite dimensional cases corresponding to being separable having a countable dense subset or not.The basic vocabulary and fundamental results of general spectral theory of banach algebras over c.

The results concering the spectrum of compact operators on a Hilbert space and add a few important facts, such as the definition and standard properties of trace-class operators.

As a general rule ,wecan not except even for self-adjoint operators to have a spectral theory built entirely on eigenvalues.The case of operators for which the eigenvalues essentially suffice to describe the spectrum those are compact operators.

The compact operators to the class of normal bounded operators TϵL(H).it may will not have sufficiently many eigenvalues to use only the point spectrum to describe it up to equivalence.The spectrum represents an operator σ and its action on vectors ϑϵH.

The spectral theorem and the continuous functional calculus to normal operators. There is a class of bounded linear transformation on a Hilbert space H that is clearly analogous to linear transformation between finite dimensional the compact operators. A banach algebra is a normed algebra which is complete as a normed space.A normed algebra is a normed vector space and an algebra.

BASIC DEFINITIONS:

HILBERT SPACE:

A complex inner produt space is a complex vector space X together with an inner product space :a function from XxX into c satisfying:

1.$(\forall x \epsilon X) <x,x> \geq 0; <x,x>=0$ iffx=0.

2.$(\forall \alpha,\beta \epsilon C)(\forall_{x,y,z} \epsilon X), <\alpha x+\beta y,z>=\alpha<x,z>+\beta<y,z>$.

3.$(\forall_{x,y} \epsilon x)\ <y,x>=\overline{<x,y>}$

INNER PRODUCT SPACE:

Suppose V is a inner product space. Aliner transformation SϵL(v) is self adjoint if

$<S_v,W>=<V,S_w>$ (v,wϵV).

The point of these notes is to explain a proof.

CYCLIC:

Let A be a set of operators in L(H).A vector xϵH is cyclic for A if $A\epsilon:=\{T_x:T\epsilon A\}$ is dense in H.

SPECTRAL MEASURE:

BANACH ALGEBRA:

A spectral measure on a Hilbert space H is a mapping and from the σ-algebra of borel measurable subsets of c into L(H) so that,

a)ϵ is a finitely additive, multiplication measure with values in L(H).

i.e)$\epsilon(A \cup B)$=L(H). if $A_o B=\varphi$ and $\epsilon(A \cup B)=\epsilon(A)\epsilon(B)$ for borel sets A,B.

b)ϵ(A) is an orthogonal projection for each borelsetA.

c)ϵ is strongly σ-additive. i.e) for each yϵH, the measure $\epsilon_A(\dagger)=\epsilon(A)(\dagger)$ with values in H is σ-additive.

i.e)if A is the disjoint union of the measurable sets A_n then $\epsilon(\cup A_1)\dagger=\sum\epsilon(A_1\backslash(Y))$.

Definition:

A normal algebra is a normal space A_o, with the following additional structure, there exist a well –defined multiplication in A_o, meaning a map $A_o X A_o \rightarrow A_o$, denoted by $(x,y)\rightarrow xy$, which satisfying the following conditions, for all $x,y,z \epsilon A_o$, $\alpha \epsilon \mathrm{Ȼ}$:

1.(associativity) (xy)z=x(yz).

2.(distributivity) $(\alpha x+y)z=\alpha xz+yz, z(\alpha x+y)=\alpha zx+zy$.

3.(sub-multiplicativity of norm)$\|xy\|\leq\|x\|.\|y\|$.

Abanach algebra is a normed algebra which is complete as a normed space .A normed algebra is said to be unital if it has a multiplicative identity.

i.e)If there exists an element which we shall denote simply by,1,

such that,

$|x=x|=x\ \forall x$.

DEFINITION:

Let A be a unit banach algebra ,and let xϵA.

Then, the spectrum of x is the set, denoted by σ(x),defined by σ(x)={λCϬ: (x-λ) is not invertible} and the spectral radius of x is defined by,

r(x)=sup{|λ|:λϵσ(x)}.

DEFINITION:

Let X and Y be banach spaces. A linear operator C:X→Y is said to be compact if for each bounded sequence $\{x_i\}$ iϵNCX.

There is a subsequence of $\{Cx_i\}$ iϵN that is convergent.

EX:

Let a<b and c<d.

If c:[c,d]x[a,d]→c is continuous, then the integral operator

$(cf)(y)=\int_a^b c(y,x)f(x)dx$

Is compact as an operator from x=c[a,b].The space of continuous functions on [a,b] with supremum norm to y=c[c,d].

I BANACH ALGEBRA AND COMPACT OPERATORS.

THEOREM: SPECTRAL THEOREM FOR COMPACT OPERATORS.

Let H be an infinite dimensional Hilbert space,and let TϵK(H) be a compact operator.

(1)Except for the possible value 0, the spectrum of T is entirely point spectrum; in other words

$\sigma(T)-\{0\}=\sigma_p(T)-\{0\}$.

(2)We have $0\epsilon\sigma(T)$, and $0\epsilon\sigma_p(T)$ if and only if T is not injective.

(3)The point spectrum outside of 0 is countable and has finite multiplicity: for each $\lambda\epsilon\sigma_p(T)-\{0\}$, we have

$\dim Ker(\lambda-T)<+\infty$.

(4)Assume T is normal. Let H_0=Ker(T), and $H_1=Ker(T)^{\perp}$.Then T maps $H_0$to H_1on H_1; on H_1, which is separable, there exist an orthonormal basis($e_1,e_2....e_n....$)and $\lambda_n\epsilon\sigma_p(T)-\{0\}$. Such that

$\lim_{n\to+\infty} \lambda_n=0$, and

$T(e_n)=\lambda_n e_n$ for all n≥1.

In particular,if $(f_i)_{i\epsilon I}$ is an arbitrary basis of H_0, which may not be separable ,we have

$T(\sum_{i\epsilon I}\alpha_i f_i+\sum_{n\geq 1}\alpha_n e_n)=\sum_{n\geq 1}\lambda_n\alpha_n e_n$

For all scalars α_i, $\alpha_n \epsilon C$ for which the vector on the left-hand side lies in H, and the series on the right convergent in H. This can be expressed also as

$T(v)=\sum_{n\geq 1}\lambda_n\langle v,e_n\rangle e_n$.

In example, We will prove the self-adjoint case using the general spectral theorem for bounded self-adjoint operators.

Will be the most commonly used version of this statement for T a normal compact operator, where the(e_n) form an orthonormal basis of $Ker(T)^{\perp}$ and $T(e_n)=\lambda_n e_n$.

II SPECTRAL THEORY FOR BOUNDED OPERATORS.

THEOREM: Let H be a separable Hilbert space and TϵL(H) a continuous normal operator. There exists a finite measure space (X,μ), a unitary operator

$$U:H\rightarrow L^2(X,\mu)$$

And a bounded function $g\epsilon L^{\infty}(X,\mu)$, such that

$M_{g^\circ}U=U_\circ T$,

Or in other words, for all $f\epsilon L^2(X,\mu)$, we have

$$(UTU^{-1})f(x)=g(x)f(x),$$

For (almost) all xϵX.

THEOREM: CONTINUOUS FUNCTIONAL CALCULUS.

Let H be a Hilbert space and TϵL(H) a self-adjoint bounded operator. There exists a unique linear map

$$\Phi=\varphi_T:C(\sigma(T))\rightarrow L(H),$$

Also denoted f→f(T), with the following properties:

-(0) This extends naturally the definition above for polynomials, i.e., for any pϵC[X] as before, we have

$\Phi(p)=p(T)=\sum_{j=0}^{d}\alpha(j)T^j$.

-(1) This map is a Banach-algebra isometric homomorphism,i.e.,we have

$$\Phi(f_1f_2)=\varphi(f_1)\varphi(f_2) \text{ for all } f_i\in c(\sigma(T)),\quad \varphi(Id)=Id, \text{and}$$

$$\|\varphi(f)\|=\|f\|C(\sigma(T)).$$

In addition, this homomorphism has the following properties:

(2)For any fϵC(σ(T)), we have $\varphi(f)^*=\varphi(\bar{f})$, i.e., $f(T)^*=\bar{f}(T)$, and in particular f(T) is normal for all fϵC(σ(T)).In addition

$$f\geq 0\Longrightarrow\varphi(f)\geq 0.$$

(3)If λϵσ(T) is in the point spectrum and vϵKer(λ-T), then vϵKer(f(λ)-f(T)).

(4)More generally, we have the spectral mapping theorem:

$$\Sigma(f(T))=f(\sigma(T))=\sigma(f), \text{ where } \sigma(f) \text{ is computed for } f\epsilon C(\sigma(T)).$$

SPECTRAL MEASURES

PROPOSITION:Let H be a Hilbert space, let T∊L(H) be a self-adjoint operator and let v∊H be a fixed vector. There exists a unique positive Radon measure μ on σ(T),depending on T and v, such that

$\int_{\sigma(T)} f(x)d\mu(x)=\langle f(T)v,v\rangle$

for all f∊C(σ(T)).In particular ,we have

$\mu(\sigma(T))=\|V\|^2$,

So μ is a finite measure.

This measure is called the spectral measure associated to v and T.

PROOF:

This is a direct application of the Riesz-MarkocTheorem;indeed,we have the linear functional

$\ell\{C(\sigma(T))\rightarrow C, f\rightarrow\langle f(T)v,v\rangle$

This is well-defined and position,since if f≥0, we have f(T)≥0,hence ⟨f(T)v,v⟩≥0 by definition.By the Riesz-Markoctheorem,therefore, there exists a unique Radon measure μ on σ(T) such that

$\ell(f)=\langle f(T)v,v\rangle=\int_{\sigma(T)} f(X)d\mu(x)$

for all f∊C(σ(T)).

Moreover, taking f(x)=1 for all x.

III .SPECTRAL THEOREM FOR NORMAL OPERATORS

LEMMA:

Let H be a Hilbert space and T∊L(H) a normal bounded operator. There exist two self-adjoint operators $T_1,T_2 \in L(H)$ such that $T=T_1+iT_2$, and $T_1T_2=T_2T_1$.

PROOF:

Write $T_1=\frac{T+T*}{2}$, $T_2=\frac{T-T*}{2i}$,

So that $T=T_1+iT_2$, and observe first that both are obviously self-adjoint,and then that

$T_1T_2=T_2T_1=T^2-(T^*)^2/4i$

Because T is normal.

PROPOSITION:

Let H be a separable Hilbert space and let T∊L(H) be a normal bounded operator.There exist a finite measure space (X,μ),a bounded measurable function g∊$L^\infty(X,\mu)$ and a unitary isomorphism

$U:H\rightarrow L^2(X,\mu)$

Such that $M_g\circ U=U\circ T$.

PROOF:

Write $T=T_1+iT_2$ with T_1, T_2 both self- adjoint bounded operators which commue, as in the lemma .Let ∏ denote the projection valued measure for T_1.The idea will be to first construct a suitable projection valued measure associated with T, which must be defined on C since σ(T) is not a subset of R.

We first claim that all projection $\prod_{1,A}$ and $\prod_{2,B}$ commute; this is because $T_1$and T_2 commute. This allows us to define

$$Й_{AXB}=\prod_{1,A}\prod_{2,B}=\prod_{2,B}\prod_{1,A},$$

Which are orthogonal projections.By basic limiting procedures,one shows that the mapping

$$AXB\rightarrow Й_{AXB}$$

Extends to a map

$$ß(C)\rightarrow P(H)$$

Which is a projection valued measure defined on the Borelsubseta of C, the definition of which is obvious. Repeating section allows us to define normal operators

$\int_C f(\lambda)dЙ(\lambda)\in L(H)$,

for f bounded and measurable defined on C .In particular ,one finds again that

$$T=\int_c \lambda dЙ(\lambda),$$

Where the integral is again defined by truncating outside a sufficiently large compact set.

So we get the spectral theorem for T ,expressed in the language of projection-valued measure.

Next one gets ,forf∈C(σ(T))and v∈H,the fundamental relation

$$\| (\int f dЙ)v \| = \int | f | d\mu_{,v}$$

Where μ_v is the associated spectral measure .This allows,again ,to show when T has a cyclic vector v(defined now as a vector for which the span of the vectors $T^n v,(T^*)^m v$, is dense),the unitary map

$L^2(\sigma(T),\mu_v).\rightarrow H$

Represents T as a multiplication operator M_z on $L^2(\sigma(T),\mu_v)$. And then Zorn's lemma allows us to get the general case.

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