A note on Linear codes associated to Schubert varieties

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Abstract:
We consider the linear code associated with Schubert subvariety of Grassmannian’s. In this review article we have studied the basic notions of Schubert code $C_{\alpha}(l, m)$. We have discussed some known results and examples of Schubert code.

Key Words: Linear code, Grassmannian, MDC, Schubert

1. Introduction:
Let $q$ be the power of a fixed prime and $l$ and $m$ be the positive integers with $l \leq m$. Let $F_q$ denote the finite field with $q$ elements and $F_q^m$ be the vector space of dimension $m$ over $F_q$. Let $\binom{m}{l}_q$ denote Gaussian binomial coefficient and $G_{l,m}$ denote the Grassmannian of all $l$-planes of $F_q^m$. It is also known that the Grassmannian $G_{l,m}$ embeds into projective space $P(F_q)\left[{\binom{m}{l}}\right]^{-1} = P(\binom{m}{l})^{-1}$ via Plücker embedding. The image under this mapping is a projective algebraic variety. To every projective space one can associate a linear code (example in [1]). The linear code corresponding to $G_{l,m}$ is known as Grassmannian code and it is denoted by $C(l, m)$. The Grassmannian code were introduced by C. T. Ryan in [2,3] for binary case. D Yu Nogin in [4] studied the linear code $C(l, m)$ associated to the Grassmannian $G_{l,m}$ over finite field and verified that $C(l, m)$ is an $[n, k, d]_q$ code where

$$n = \binom{m}{l}_q = \frac{(q^{m-1})(q^{m-2}) \cdots (q^{m-l})(q^{m-l+1}) \cdots (q^{l-1})(q^{l-2}) \cdots (q^{l-1})}{(q^{l-1})(q^{l-2}) \cdots (q^{l-1})}, \quad k = \binom{m}{l} \quad \text{and} \quad d = q^{l(m-l)}$$

For fix integers $k, n$ with $1 \leq k \leq n$ and prime $q$. Let $C$ be linear $[n, k]_q$ code i.e $C$ is $k$-dimensional subspace of $F_q^n$. Given for any $x = (x_1, x_2, \ldots, x_n) \in F_q^n$, define $\text{sup}(x) = \{i : x_i \neq 0\}$ and $\|x\| = \text{sup}(x)$. The image under this mapping is a projective algebraic variety.
|sup(x)| denote support and Hamming norm of x, likewise for \( D \subseteq F_q^n \) \( \text{sup}(D) = \{ i : x_i \neq 0 \ \text{for some} \ x = (x_1, x_2, \ldots, x_n) \in D \} \) and \( \|D\| = |\text{sup}(D)| \) denote support and hamming norm of D. The minimum distance or hamming weight of C is defined by \( d(C) := \min\{||x|| : x \in C \text{ with } x \neq 0\} \).

For some positive integer \( r \), the \( r \)-th higher weight or \( r \)-th generalized hamming weight is denoted by \( d_r(C) \) of the code \( C \) defined by
\[
d_r(C) = \min\{|\text{sup}(D)| : D \text{ is subspace of } C \text{ and } \dim(D) = r \}
\]
The set \( \{d_r(C) : 1 \leq r \leq k \} \) is complete weight hierarchy of the code \( C \). For linear code \( C \) it is very interesting and difficult to determine complete weight hierarchy.

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \) be strictly increasing sequence of positive integers satisfying \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_l \leq m \) and \( \Omega_{\alpha}(l, m) \) be corresponding Schubert variety in Grassmannian \( G_{l,m} \). Schubert varieties are sub-varieties of Grassmannian. Likewise Grassmannian varieties, the Schubert variety may considered as subset of \( P^{(m)}_{(l)^{-1}} \). The linear code corresponding to Schubert variety \( \Omega_{\alpha}(l, m) \) is called Schubert code and this code is denoted by \( C_{\alpha}(l, m) \). Ghorpade-Tsfasman [5] had proved the minimum distance conjecture for Schubert code corresponding to case when \( \delta(\alpha) = l(m - l) - 1 \) and found length and dimension od Schubert code in general. In [5] Ghorpade -Tsfasman proved that Schubert code \( C_{\alpha}(l, m) \) is \([n_{\alpha}, k_{\alpha}] \) code where
\[
n_{\alpha} = \sum_{\beta \leq \alpha} q^{\delta(\beta)}, \quad k_{\alpha} = \det_{1 \leq i, j \leq l} \left( \binom{\alpha_j - j + 1}{i - j + 1} \right)
\]
where sum is run over all \( l \)-tuples \( \beta = (\beta_1, \beta_2, \ldots, \beta_l) \) of integers satisfying \( 1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_l \leq m \) and \( \beta_i \leq \alpha_i \) for \( i = 1, 2, 3 \ldots l \) and \( \delta(\beta) = \sum_{i=1}^{l} (\beta_i - i) \). Alternating proof of MDC for Schubert code was given in [6] and [7].

2. Preliminaries :

The Grassmannian of all \( l \)-planes of \( F_q^m \) is given by,
\[
G_{l,m} := \{ L \subseteq F_q^m : L \text{ is subspace of } F_q^m \text{ and } \dim(L) = l \}.
\]
Due to plucker embedding of \( G_{l,m} \) into projective space \( P^{(m)}_{(l)^{-1}} \), choose matrix \( A_L \) whose rows forms basis for \( L \). The order of \( A_L \) will be \( l \times m \).
matrix with rank $l$. In $A_L$ we have $\binom{m}{l}$ minors of order $l$. By fixing some ordering of these minors and map $L$ onto $\binom{m}{l}$ tuples of minors of $A_L$ of size $l$ which is required embedding of $G_{l,m}$ into projective space $p_{l}^{\binom{m}{l}}$.

Let $X = (X_{ij})$ be $l \times m$ indeterminate matrix over $F_q$. Let $l$-multiset $I \subset \{1,2, ..., m\}$ denote $l \times l$ minor of $X$ corresponding to the columns indexed by $I$ by $\text{det}_I(X)$ and $F_q[X]_l$ be a vector space over $F_q$ spanning by minors of $\text{det}_I(X)$. Then for any $L$ in $G_{l,m}$, the $l \times m$ matrix $A_L$ is matrix whose rows span $L$.

Consider the evaluation map $Ev : F_q[X]_l \rightarrow F_q^{\binom{m}{l}}$ defined by $f = \sum_{I} \alpha_I \text{det}_I(X) \rightarrow (f(A_L))_{L \in G_{l,m}}$ and $f(A_L) = \sum_{I} \alpha_I \text{det}_I(A_L)$ i.e. $f(A_L)$ is evaluation of $f$ at $A_L$. The image of this is called Grassmann code $C(l,m)$. The image $Ev(f)$ of $f$ denoted by $c_f$ is codeword corresponding to $f$.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$ be the sequence of strictly increasing positive integers and $A_1 \subset A_2 \subset \ldots \subset A_l$ be the sequence of subspaces of $F_q^m$ with $\dim A_i = \alpha_i \forall i$.

Let $\Omega_{\alpha}(l,m) = \{W \in G_{l,m} : \dim(W \cap A_i) \geq i, \forall i\}$ be the Schubert variety in $G_{l,m}$ corresponding to sequence $\alpha$. If the above evaluation map is restricted to $\Omega_{\alpha}(l,m)$ then it will be Schubert code $C_{\alpha}(l,m)$.

The Schubert variety $\Omega_{\alpha}(l,m)$ only depends on the sequence $\alpha$ not on corresponding sequence $A_1 \subset A_2 \subset \ldots \subset A_l$ of subspaces. If $B_1 \subset B_2 \subset \ldots \subset B_l$ is another sequence of subspaces of $F_q^m$ with $\dim B_i = \alpha_i$ for every $i$. If $\Omega_{\alpha}(A_1, A_2, ..., A_l, m)$ denotes the Schubert varieties corresponding to the sequence of subspaces $A_1 \subset A_2 \subset \ldots \subset A_l$ and $\Omega_{\alpha}(B_1, B_2, ..., B_l, m)$ denotes Schubert variety corresponding to sequence of subspaces $B_1 \subset B_2 \subset \ldots \subset B_l$ then due to [1] these corresponding Schubert codes are equivalent.

3. Basic Notions and some known facts about Schubert Code:

For fix integer $l$, $m$ with $1 \leq l \leq m$. Let $I(l,m)$ be the indexing set with partial order $\leq$ for any $\beta = (\beta_1, \beta_2, ..., \beta_l) \in I(l,m)$.

Let $\delta_\beta = \sum_{i=1}^{l} (\beta_i - i) = \beta_1 + \beta_2 + \cdots + \beta_l - \frac{l(l+1)}{2}$
For fix some $\alpha \in I(l, m)$ then $C_\alpha(l, m)$ be the corresponding linear code and length of Schubert code [9] is $n_\alpha$ given by

**Thm 1** (Length of Schubert Code )[9]The number of $F_q$ rational points of $\Omega_\alpha$ which also length $n_\alpha$ of $C_\alpha(l, m)$ is given by

$$n_\alpha = \sum_{\beta \leq \alpha} q^\delta\beta$$

where the sum is over all $\beta \in I(l, m)$ satisfying $\beta \leq \alpha$

With the previous notations the explicit formula for the dimension of Schubert code $C_\alpha(l, m)$ is $K_\alpha$ is given by [9] in following result .

**Thm 2** (Dimension of Schubert Code )

The dimension of Schubert code $C_\alpha(l, m)$ is given by $l \times l$ determinant and given due to [9] by

$$K_\alpha = det_{1 \leq i, j \leq l} \left( \begin{array}{cccc}
(\alpha_1 - j + 1) & 1 & 0 & \cdots \\
(\alpha_2 - j + 1) & 1 & \cdots & 0 \\
(\alpha_3 - j + 1) & \cdots & \cdots & \cdots \\
(\alpha_l - j + 1) & \cdots & \cdots & 1 \\
\end{array} \right)$$

**Thm 3**[9]: The dimension $k_\alpha$ of q-ary Schubert code $C_\alpha(l, m)$ is independent of $q$ and is related to the length $n_\alpha = n_\alpha(q)$ of $C_\alpha(l, m)$ by the formula

$$\lim_{q \to 1} n_\alpha(q) = k_\alpha$$

4. Examples of Schubert Code :

**Example 4.1**

Let $\alpha = (2,4)$ then $C_\alpha(2,4)$ is corresponding Schubert code then dimension of $C_\alpha(2,4)$ is 5

**Example 4.2**

Let $\alpha = (3,5)$ then $C_\alpha(3,5)$ is corresponding Schubert code and dimension of $C_\alpha(3,5)$ is 9

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5. References :


