SOLUTION OF PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS BY USING MOHAND AND DOUBLE MOHAND TRANSFORM METHODS

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Abstract: Partial integro-differential equations (PIDE) occur in several fields of sciences and mathematics. The main purpose of this paper is to study how to solve partial integro-differential equation (PIDE) by various methods like Mohand and Double Mohand Transform. To solve PIDE by using Mohand Transform (MT), first convert proposed PIDE to an ordinary differential equation (ODE) then solving this ODE by applying inverse MT we get an exact solution of the problem. To solve PIDE by using Double Mohand Transform (DMT), first convert proposed PIDE to an algebraic equation, solving this algebraic and applying double inverse Mohand Transform, we obtain an exact solution of the problem.

Index Terms - Partial integro-differential equations (PIDE), ordinary differential equation (ODE), Mohand Transform (MT), Double Mohand Transform (DMT).

I. INTRODUCTION

In the last few years theory and application of partial integro-differential equations (PIDE) play an important role in various fields of many problems of mathematical fields, engineering physics, biology, and social sciences [4-12]. This explains a growing interest in the mathematics community to integro-differential equations and in particular to partial integro-differential equations. Therefore, it is very important to know various methods to solve such partial differential equations [1,2,3].

In this paper, we solve examples of PIDE by using two different methods like Mohand and Double Mohand Transform.

II. PRELIMINARIES

2.1 Mohand Transform:

Definition:
Mohand transform of the function \( f(t) \) is defined as

\[
M[f(t)] = R(v) = v^2 \int_0^\infty f(t) e^{-vt} dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2
\]

Theorem 1:
Mohand transform of partial derivatives are in the form:

\[
M\left[ \frac{\partial f(x,v)}{\partial x} \right] = vR(x,v) - v^2 f(x,0) \quad M\left[ \frac{\partial^2 f(x,v)}{\partial x^2} \right] = v^2 R(x,v) - v^3 f(x,0) - v^2 \frac{\partial f(x,0)}{\partial t} \\
M\left[ \frac{\partial f(x,v)}{\partial t} \right] = \frac{d}{dx} [R(x,v)] \quad M\left[ \frac{\partial^2 f(x,v)}{\partial x^2} \right] = \frac{d^2}{dx^2} [R(x,v)]
\]

Theorem 2: (Convolution):
Let \( f(t) \) and \( g(t) \) having Mohand transform \( M(v) \) and \( (v) \), then Mohand transform of the convolution \( f \) and \( g \), \( f(t) * g(t) = \int_0^\infty f(t) g(t - \tau) d\tau \), is given by \( M[ f(t) * g(t) ] = \frac{1}{\sqrt{2\pi}} M(v) N(v) \)

Solving PIDEs using Mohand Transform Method:

Consider general linear PIDE,

\[
\sum_{i=0}^m a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^n b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^r d_i \int_0^t k_i(t-s) \frac{\partial^i u(x,s)}{\partial s^i} + f(x,t) = 0
\]

(With prescribed condition).
Using theorem 1 and theorem 2 for Mohand transform, we get
\[\sum_{i=0}^{n} a_i [v^i R(x,v) - v^{i+1} u(x,0) - \cdots - u_{t(i+1)}(x,0)] + \sum_{i=0}^{n} b_i \frac{d^i}{dx^i} R(x,v) + c R(x,v) + \sum_{i=0}^{n} d_i \bar{u}_i(v) \frac{d^i}{ds^i} R(x,v) = f(x,v) = 0\]  \hspace{1cm} (2)
Where \(M[u(x,t)] = R(x,v)\), \(M[u_i(t)] = \bar{u}_i(v)\) and \(M[f(x,t)] = \bar{f}(x,v)\).
Solving this ODE (2) and taking inverse Aboodh transform of \(R(u,v)\), we get solution \(u(x,t)\) of (1)

**Illustrative example:**

**Example 1:**
Consider the PIDE,
\[\frac{d^2 u}{dx^2} = \frac{\partial^2 u}{\partial x \partial t} + 2 \int_0^t (t-s) u(x,s) ds - 2e^x\]  \hspace{1cm} (3)
with initial condition \(u(x,0) = e^x\), \(u_t(x,0) = 0\) and boundary condition \(u(0,t) = \text{cost}\)
Taking Mohand transform of Eq. (3), we have
\[M[u_{xx}] = M[u_x] + 2M \left[ \int_0^t (t-s) u(x,s) ds \right] - 2e^xM[1]\]
\[v^2 R(x,v) - v^2 u(x,0) - v^2 u_t(x,0) = \frac{d}{dx} R(x,v) + \frac{2}{v^2} M[1]M[u(x,s)] - 2e^xM[1]\]
\[R'(x,v) + \left( \frac{2}{v^2} - v^2 \right) R(x,v) = (2v - v^3) e^x\]
\[R(x,v) = \frac{v^3}{v^2+1} e^x + c e^{(v^2 - 2x)/v^2}\]  \hspace{1cm} (4)
Now, \(M[u(0,t)] = R(0,v) = M[\text{cost}] = \frac{v^3}{v^2+1}\) \hspace{1cm} (5)

Compare (4) and (5), we get \(c = 0\)

Equation (4) implies,
\[R(x,v) = e^x \frac{v^3}{v^2+1}\]
Applying inverse Mohand transform on both sides,
\[u(x,t) = e^x \text{cost}\]  \hspace{1cm} (6)

**Example 2:**
Consider the linear partial integro- differential equation,
\[u_t - u_{xx} + u + \int_0^t e^{t-s} u(x,s) ds = (x^2 + 1)e^t - 2\]  \hspace{1cm} (7)
\[u(x,0) = x^2, \quad u_t(x,0) = 1\]
\[u(0,t) = t, \quad u_t(0,t) = 0\]
Taking Mohand transform of Eq. (7), we have
\[M[u_t] - M[u_{xx}] = M[u] + M \left[ \int_0^t e^{t-s} u(x,s) ds \right] = (x^2 + 1)M[e^t] - 2M[1]\]
\[vR(x,v) - v^2 u(x,0) - \frac{d^2 R}{dx^2} + R + \frac{v^2 R}{(v-1)v^2} = (x^2 + 1)\frac{v^2}{v-1} - 2v\]
\[- \frac{d^2 R}{dx^2} + \frac{v^2 R}{(v-1)} = \frac{-v^2}{v-1} x^2 + \frac{v^2 - 2v}{x-1}\]  \hspace{1cm} (8)
Solving (8), we get
\[R(x,v) = c_1 e^{\frac{v^2}{v-1}} + c_2 e^{\frac{-v^2}{v-1}} + x^2 v + 1\]  \hspace{1cm} (9)
Now, \(u(0,t) = t\),
so \(R(0,v) = M[u(0,t)] = M[t] = 1\) \hspace{1cm} (10)
Using (9) and (10), we get \(c_1 + c_2 = 0\) \hspace{1cm} (11)
And \(c_1 - c_2 = 0\) \hspace{1cm} (12)
Solving (11) and (12), we get \(c_1 = c_2 = 0\) \hspace{1cm} (13)

Then equation (9) becomes
\[R(x,v) = x^2 v + 1\]
Applying inverse Mohand transform on both sides,
\[u(x,t) = x^2 M^{-1}[v] + M^{-1}[1]\]
\[= x^2 + t\]

\[\sum_{i=0}^{n} a_i [v^i R(x,v) - v^{i+1} u(x,0) - \cdots - u_{t(i+1)}(x,0)] + \sum_{i=0}^{n} b_i \frac{d^i}{dx^i} R(x,v) + c R(x,v) + \sum_{i=0}^{n} d_i \bar{u}_i(v) \frac{d^i}{ds^i} R(x,v) = f(x,v) = 0\]
2.2 Double Mohand Transform:

Definition:
Let \( f(x, t) \), where \( t, x \in R^+ \) be a function, which can be expressed as a convergent infinite series then, its double Aboodh transform given by:

\[
M_2[f(x, t), u, v] = R(u, v) = u^2v^2 \int_0^\infty \int_0^\infty f(t, x)e^{-(ux+vt)}dx \, dt, \ x, t \geq 0, \ \text{where} \ u \ \text{and} \ v \ \text{are complex values.}
\]

Theorem 1:
Double transform of first Mohand and second order partial derivatives are in the form:

\[
M_2[f_t] = uR(u, v) - u^2R(0, v) \quad M_2[f_{xx}] = u^2R(u, v) - u^3R(0, v) - u^2 \frac{\partial}{\partial x} R(0, v)
\]

\[
M_2[f_v] = vR(u, v) - v^2R(0, v) \quad M_2[f_{xx}] = v^2R(u, v) - v^3R(0, v) - v^2 \frac{\partial}{\partial t} R(0, v)
\]

\[
M_2[f_{xv}] = u^2v^2f(0, 0) - u^2R(u, v) - \frac{u^2}{v} R(0, v)
\]

Proof:

\[
M_2[f_t] = u^2v^2 \int_0^\infty \int_0^\infty f(x, t)e^{-(ux+vt)}dx \, dt, \ x, t > 0
\]

\[
M_2 \left[ \frac{\partial f}{\partial x} \right] = u^2v^2 \int_0^\infty \int_0^\infty \frac{\partial f}{\partial x} e^{-(ux+vt)}dx \, dt
\]

\[
= u^2v^2 \int_0^\infty e^{-vt} \left[ u^2 \int_0^\infty e^{-ux} \frac{\partial}{\partial x} f(x, t)dx \right] dt
\]

\[
= u^2v^2 \int_0^\infty e^{-vt} \left[ u^2 f(0, t) + u^2 \int_0^\infty e^{-ux} \frac{\partial f(x, t)}{\partial x} dx \right] dt + u^2v^2 \int_0^\infty e^{-vt} e^{-ux} f(x, t) dx dt
\]

\[
M_2[f_{xx}] = -u^2R(0, v) + uR(u, v)
\]

\[
M_2 \left[ \frac{\partial^2 f}{\partial x^2} \right] = u^2v^2 \int_0^\infty \int_0^\infty e^{-(ux+vt)} \frac{\partial^2 f}{\partial x^2} dx \, dt
\]

\[
= v^2 \int_0^\infty e^{-vt} \left[ -u^2 \frac{\partial}{\partial x} f(0, t) + \frac{\partial}{\partial x} f(x, t) \right] dt
\]

\[
= -u^2 \left[ v^2 \int_0^\infty e^{-vt} \frac{\partial}{\partial x} f(0, t) dt + \frac{\partial}{\partial x} f(x, t) dx \right] + u^2v^2 \int_0^\infty e^{-vt} e^{-ux} \frac{\partial^2 f(x, t)}{\partial x^2} dx dt
\]

\[
= -u^2 \frac{\partial}{\partial x} R(0, v) + u[uR(u, v) - u^2R(0, v)]
\]

Theorem 2 (Convolution):
Let \( f(x, t) \) and \( g(x, t) \) be the functions having Double Mohand transform \( N_1(u, v) \) and \( N_2(u, v) \), then the Double Mohand transform of convolution of \( f(x, t) \) and \( g(x, t) \) is,

\[
M_2[(f \ast g)(x, t); (u, v)] = \frac{1}{u^2v^2} N_1(u, v)N_2(u, v)
\]

Solving PIDE’s using Double Mohand transform Method:
Consider general linear PIDE,

\[
\sum_{n=0}^{m} a_n \frac{\partial^n u(x, t)}{\partial t^n} + \sum_{n=0}^{N} b_n \frac{\partial^n u(x, t)}{\partial t^n} + cu + \sum_{i=0}^{N} d_i \int_0^t k_i(t-s) \frac{\partial u(x, s)}{\partial x} \, ds + f(x, t) = 0
\]

(With prescribed condition).

Using theorem 1 and theorem 2 for double Mohand transform, we get
\[\sum_{i=0}^{m} a_i M_2 \left[ \frac{\partial u(x,t)}{\partial t} \right] + \sum_{i=0}^{n} b_i M_2 \left[ \frac{\partial^2 u(x,t)}{\partial s^2} \right] + c M_2[u] + \sum_{i=0}^{r} d_i M_2 \left[ \int_0^t k_i(t-s) \frac{\partial u(x,s)}{\partial x} \right] + M_2[f(x,t)] = 0\]

\[\sum_{i=0}^{m} a_i \left\{ v^i R(u,v) - \frac{v^{i-1}}{2} \frac{\partial}{\partial v} R(x,0) \right\} + \sum_{i=0}^{n} b_i \left\{ u^i R(u,v) - \frac{u^{i-1}}{2} \frac{\partial}{\partial u} R(0,t) \right\} + c R(u,v) + \sum_{i=0}^{r} d_i \int_0^t k_i(t-s) R(u,v) \left[ u^i R(u,v) - \frac{u^{i-1}}{2} \frac{\partial}{\partial u} R(0,t) \right] + \tilde{R}(u,v) = 0\]  

(15)

Where \(M_2[u(x,t)] = R(x,v), M_2[k_i(t)] = \tilde{k}_i(v)\) and \(M_2[f(x,t)] = \tilde{f}(x,v)\).

After solving this ordinary differential equation and taking inverse double Mohand transform of \(R(x,v)\), we have the required solution \(u(x,t)\).

**Illustrative example:**

**Example 1:**

Consider the PIDE,

\[u_{tt} = u_x + 2 \int_0^t (t-s) u(x,s)ds - 2e^x\]

with initial condition \(u(x, 0) = e^x\), \(u_t(x,0) = 0\) and boundary condition \(u(0,t) = \cos t\).

Taking double Mohand transform of Eq. (16), we have

\[M_2[u_{tt}] = M_2[u_x] + 2M_2 \left[ \int_0^t (t-s) u(x,s)ds \right] - 2M_2[e^x]\]

\[v^2 R(u,v) - u^2 R(0,v) - u^2 R(u,0) = uR(u,v) - u^2 R(0,v) + \frac{2}{v^2} R(u,v) - \frac{2u^2v}{v^2 - u^2}\]

(19)

Moreover, single Mohand transform of initial conditions (17) & boundary condition (18) are given by:

\[R(u,0) = \frac{u^2}{u^2 - 1}, R_t(u,0) = 0, R(0,v) = \frac{v}{v^2 + 1}\]

Then equation (19) becomes

\[uR(u,v) + \frac{2}{v^2} R(u,v) - \frac{u^2v}{(v^2 + 1)(v^2 - 1)} R(u,v) = \frac{u^2v}{v^2 + 1} + 2u^2v\frac{u^2 - 1}{v^2 - u^2} \frac{u^2 - 3}{(v^2 + 1)(v^2 - 1)} \frac{u^2 - 3}{u^2 - 1}\]

(20)

Applying inverse double Mohand transform of (20), we get

\[u(x,t) = e^x \cos t\]

**Example 2:**

Consider the PIDE,

\[u_t - u_{xx} + u + \int_0^t e^{s-x} u(x,s)ds = (x^2 + 1)e^t - 2\]

\[u(x,0) = x^2, u_t(x,0) = 0\]

\[u(0,t) = t, u_x(0,t) = 0\]

Taking double Mohand transform of Eq. (21), we have

\[M_2[u_t] - M_2[u_{xx}] + M_2[u] + M_2 \left[ \int_0^t e^{s-x} u(x,s)ds \right] = M_2[(x^2 + 1)e^t] - M_2[2]\]

\[uR(u,v) - u^2 R(u,v) + \frac{1}{v^2} R(u,v) - \frac{1}{v^2} R(u,v) = \frac{u^2+2}{u^2-1} u^2 - 2uv\]

(24)

Moreover, single Mohand transform of initial conditions (22) & boundary condition (23) are given by:

\[R(u,0) = 2, R(u,0) = v\]

\[R(0,v) = 1, R_t(0,v) = 0\]

Then equation (24) becomes

\[vR(u,v) - u^2 R(u,v) + R(u,v) + \frac{1}{v^2} R(u,v) = \frac{v^2(u^2+2)}{u^2-1} - 2uv + 2v - u^3\]

(25)

Applying inverse double Mohand transform of (25), we get

\[u(x,t) = x^2 + t\]

**Example 3:**

Consider the PIDE,
\[ u_t + u_{ttt} + \int_0^t \sinh(t - y) \ u_{xxx}(x, y) \ dy = 0 \]  
\[ u(x, 0) = 0, \quad u_x(x, 0) = x, \quad u_{tt}(x, 0) = 0 \]  
\[ u(0, t) = 0, \quad u_x(0, t) = \sin t, \quad u_{xx}(0, t) = 0 \] 

Taking double Mohand transform of Eq. (26), we have

\[ \nu R(u,v) - \nu^2 R(u,0) + \nu^3 R(u,v) - \nu^4 R(0,0) + \frac{1}{\nu^2 - 1} [u^2 R(u,v) - u^3 R(0,0) - u^2 R_x(0,v)] - \nu^3 R_x(u,0) - \nu^2 R_{ttt}(u,0) = 0 \]

Moreover, single Mohand transform of initial conditions (27) & boundary condition (28) are given by:

\[ R(u,0) = 0, \quad R_x(u,0) = \frac{\nu^2}{\nu^2 + \nu}, \quad R_{xx}(0,v) = 0 \]

Then equation (29) becomes

\[ \left( \nu + \nu^3 + \frac{\nu^2}{\nu^2 - 1} \right) R(u,v) = \nu^3 + \frac{\nu^2 \nu^2}{(\nu^2 - 1)(\nu^2 + 1)} \]

\[ R(x,v) = \frac{\nu^2}{\nu^2 + 1} \]

Applying inverse double Mohand transform of (30), we get

\[ u(x,t) = x \sin t \]

III. CONCLUSION

In this paper, we have successfully developed the double Mohand transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time.

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REFERENCES


