BASICS OF DIFFERENTIAL CALCULUS

Prof. (Dr.) Rajeshwar Prasad Yadav
Head, Department of Mathematics
R.L.S.Y.College, Bettiah, Bihar
B.R.A.Bihar University, Muzaffarpur

Abstract : In this paper we will discuss the initial value of the index in quite a few facts and theorems that we'll be seeing. We also discuss a brief look of three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however the main reason that we're going to look at two of them in this section is that they are the only types of series that we'll be looking at for which we will be able to get actual values for the series. We also discuss facts/theorems the starting point of the series will not affect the result and so to simplify the notation and to avoid giving the impression that the starting point is important we will drop the index from the notation.

Keywords : Differential, Calculus, Theorem, Special series, Notation, Starting point.

We will be dropping the initial value of the index in quite a few facts and theorems that we'll be seeing throughout this paper. In these facts/theorems the starting point of the series will not affect the result and so to simplify the notation and to avoid giving the impression that the starting point is important we will drop the index from the notation.

Do not forget however, that there is a starting point and that this will be an infinite series. Note however, that if we do put an initial value of the index on a series in a fact/theorem it is there because it really does need to be there. Now that some of the notational issues are out of the way we need to start thinking about various ways that we can manipulate series. We'll start this off with basic arithmetic with infinite series as we'll need to be able to do that on occasion. We have the following properties.

Properties

If $\sum a_n$ and $\sum b_n$ are both convergent series then,

1. $\sum ca_n$, where c is any number, is also convergent and $\sum ca_n = c\sum a_n$

2. $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$ is also convergent and, $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty}(a_n \pm b_n)$.

The first property is simply telling us that we can always factor a multiplicative constant out of an infinite series and again recall that if we don't put in an initial value of the index that the series can start at any value. Also recall that in these cases we won't put an infinite at the top either.

The second property says that if we add/subtract series all we really need to do is add/subtract the series terms. Note as well that in order to add/subtract series we need to make sure that both have the same initial value of the index and the new series will also start at this value. Before we move on to a different topic let's discuss multiplication of series briefly. We'll start both series at $n = 0$ for a later formula and then note that,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) \neq \sum_{n=0}^{\infty}(a_n b_n)$$

To convince yourself that this isn't true consider the following product of two finite sums.

$$(2+x)(3-5x+x^2) = 6-7x-3x^2+x^3$$

Yeah, it was just the multiplication of two polynomials. Each is a finite sum and so it makes the point. In doing the multiplication we didn't just multiply the constant terms, then the x terms, etc. Instead we had to distribute the 2 through the second polynomial, then distribute the x through the second polynomial and finally combine like terms.

Multiplying infinite series (even though we said we can't think of an infinite series as an infinite sum) needs to be done in the same manner. With multiplication we're really asking us to do the following.

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = (b_0 + b_1 + b_2 + b_3 + ...)$$

To do this multiplication we would have to distribute the a_0 through the second term, distribute the a_1 through etc then combine like terms. This is pretty much impossible since both series have an infinite set of terms in them, however the following formula can be used to determine the product of two series.

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n \quad \text{where } c_n = \sum_{n=0}^{\infty} a_i b_{n+1}$$

We also can't say a lot about the convergence of the product. Even if both of the original series are convergent it is possible for the product to be divergent. The reality is that multiplication of series is a somewhat difficult process and in general is avoided if possible. We will take a brief look at it towards the end of the chapter when we've got more work under our belt and we run across a situation where it might actually be what we want to do. Until then, don't worry about multiplying series. The next topic that we need to discuss in this section is that of index shift. To be honest this is not a topic that we'll see all that often in this course. In fact, we'll use it once in the next section and then not use it again in all likelihood.

Despite the fact that we won't use it much in this course doesn't mean however that it isn't used often in other classes where you might run across series. So, we will cover it briefly here so t hat you can say you've seen it. The basic idea behind index shifts is to start a series at a different value for whatever the reason (and yes, there are legitimate reasons for doing that).

Consider the following series,

$$\sum_{i=2}^{\infty} \frac{n+5}{2^n}$$

Suppose that for some reason we wanted to start this series at $n = 0$, but we didn't want to change the value of the series. This means that we can't just change the $n = 2$ to $n = 0$ as this would add in two new terms to the series and thus changing its value. Performing an index shift is a fairly simple process to do.

We'll start by defining a new index, say i, as follows,

$$i = n - 2$$

Now, when $n = 2$, we will get $i = 0$. Notice as well that if $n = \infty$ then $i = \infty - 2$, so only the lower limit will change here. Next, we can solve this for n to get,

$$n = i + 2$$

We can now completely rewrite the series in terms of the index i instead of the index n simply be plugging in our equation for n in terms of i.

$$\sum_{i=2}^{\infty} \frac{n+5}{2^n} = \sum_{i=0}^{\infty} \frac{(i+2)+5}{2^{i+2}} = \sum_{i=0}^{\infty} \frac{i+7}{2^{i+2}}$$

To finish the problem out we'll recall that the letter we used to the index doesn't matter and so we'll change the final i back into an n to get,

$$\sum_{i=2}^{\infty} \frac{n+5}{2^n} = \sum_{i=0}^{\infty} \frac{n+7}{2^{n+2}}$$

To convince yourselves that these really are the same summation let's write out the first couple of terms for each of them,

$$\sum_{i=2}^{\infty} \frac{n+5}{2^n} = \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \ldots\ldots$$

$$\sum_{i=2}^{\infty} \frac{n+7}{2^{n+2}} = \frac{7}{2^2} + \frac{8}{2^3} + \frac{9}{2^4} + \frac{10}{2^5} + \ldots\ldots$$

So, sure enough the two series do have exactly the same terms.

There is actually an easier way to do an index shift. The method given above is the technically correct way of doing an index shift. However, notice in the above example we decreased the initial value of the index by 2 and all the n's in the series terms increased by 2 as well. This will always work in this manner. If we decrease the initial value of the index by a set amount then all the other n's in the series term will increase by the same amount. Likewise, if we increase the initial value of the index by a set amount, then all the n's in the series term will decrease by the same amount. Let's do a couple of examples using this shorthand method for doing index shifts.

Example : Perform the following shifts.

1. Write $\sum_{i=1}^{\infty} arn^{-1}$ as a series that starts at $n = 0$.

2. Write $\sum_{i=1}^{\infty} \frac{n^2}{1-3^{n+1}}$ as a series that starts at $n = 3$.

Solution :

1. In this case we need to decrease the initial value by 1 and so the n's (okay the single n) in the term must increase by 1 as well.

$$\sum_{i=1}^{\infty} ar^{n-1} = \sum_{i=1}^{\infty} ar^{(n+1)-1} = \sum_{i=1}^{\infty} ar^n$$

2. For this problem we want to increase the initial value by 2 and so all the *n*'s in the series term must decrease by 2.

$$\sum_{i=1}^{\infty}\frac{n^2}{1-3^{n+1}}=\sum_{i=3}^{\infty}\frac{(n-2)^2}{1-3^{(n-2)+1}}=\sum_{i=3}^{\infty}\frac{(n-2)^2}{1-3^{n-1}}$$

The final topic in this section is again a topic that we'll not be seeing all that often in this class, although we will be seeing it more often than the index shifts.

This final topic is really more about alternate ways to write series when the situation requires it. Let's start with the following series and note that the $n = 1$ starting point is only for convenience since we need to start the series somewhere.

$$\sum_{i=1}^{\infty}a_n=a_1+a_2+a_3+a_4+a_5+.....$$

Notice that if we ignore the first term the remaining terms will also be a series that will start at $n = 2$ instead of $n = 1$. So, we can rewrite the original series as follows,

$$\sum_{n=1}^{\infty}a_n=a_1+\sum_{n=2}^{\infty}a_n$$

In this example we say that we've stripped out the first term. We could have stripped out more terms if we wanted to. In the following series we've stripped out the first two terms and the first four terms respectively.

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+\sum_{n=3}^{\infty}a_n$$

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+a_3+a_4+\sum_{n=5}^{\infty}a_n$$

Being able to strip out terms will, on occasion, simplify our work or allow us to reuse a prior result so it's an important idea to remember. Notice that in the second example above we could have also denoted the four terms that we stripped out as a finite series as follows,

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+a_3+a_4+\sum_{n=5}^{\infty}a_n=\sum_{n=1}^{4}a_n+\sum_{n=5}^{\infty}a_n$$

This is a convenient notation when we are stripping out a large number of terms or if we need to strip out an undetermined number of terms. In general, we can write a series as follows,

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{N}a_n+\sum_{n=N+1}^{\infty}a_n$$

We'll leave this section with an important warning about terminology. Don't get sequences and series confused! A sequence is a list of numbers written in a specific order while an infinite series is a limit of a sequence of finite series and hence, if it exists will be a single value. So, once again, a sequence is a list of numbers while a series is a single number, provided it makes sense to even compute the series.

Students will often confuse the two and try to use facts pertaining to one on the other. However, since they are different beasts this just won't work. There will be problems where we are using both sequences and series so we'll always have to remember that they are different.

Convergence/Divergence

In the previous section we spent some time getting familiar with series and we briefly defined convergence and divergence. Before worrying about convergence and divergence of a series we wanted to make sure that we've started to get comfortable with the notation involved in series and some of the various manipulations of series that we will, on occasion, need to be able to do.

As noted in the previous section most of what we were doing there won't be done much in this chapter. So, it is now time to start talking about the convergence and divergence of a series as this will a topic that we'll be dealing with to one extent of another in almost all of the remaining sections of this chapter. So, let's recap just what an infinite series is and what it means for a series to be convergent or divergent.

We'll start with a sequence $\{a_n\}_{n=1}^{\infty}$ and again note that we're starting the sequence at $n = 1$ only for the sake of convenience and it can, in fact, be anything.

Next we define the partial sums of the series as,

$$s_1=a_1$$

$$s_2=a_1+a_2$$

$$s_3=a_1+a_2+a_3$$

$$s_4=a_1+a_2+a_3+a_4$$

$$s_n = a_1 + a_2 + a_3 + a_4 + ...a_n = \sum_{i=1}^{\infty} a_i$$

and these form a new sequence, $\{s_n\}_{n=1}^{\infty}$.

An infinite series, or just series here since almost every series that will be looking at will be an infinite series, is then the limit of the partial sums. Or,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$$

If the sequence of partial sums is a convergent sequence (i.e. its limit exists and is finite) then the series is also called convergent and in this case if $\lim_{n\to\infty} s_n = s$ then, $\sum_{i=1}^{\infty} a_i = s$. Likewise, if the sequence of partial sums is a divergent sequence (i.e. its limit doesn't exist or is plus or minus infinity) then the series is also called divergent. Let's take a look at some series and see if we can determine if they are convergent or divergent and see if we can determine the value of any convergent series we find.

Example : Determine if the following series is convergent or divergent. If it converges determine its value.

$$\sum_{i=1}^{\infty} n$$

Solution : To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums.

$$s_n = \sum_{i=1}^{n} i$$

This is a known series and its value can be shown to be,

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Don't worry if you didn't know this formula (I'd be surprised if anyone knew it....) as you won't be required to know it in my course. So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{\frac{n(n+1)}{2}\right\}_{n=4}^{\infty}$$

Is convergent or divergent. That's not terribly difficult in this case. The limit of the sequence terms is,

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to and so the series also diverges.

So, as we saw in this example we had to know a fairly obscure formula in order to determine the convergence of this series. In general finding a formula for the general term in the sequence of partial sums in a very difficult process.

In fact after the next section we'll not be doing much with the partial sums of series due to the extreme difficulty faced in finding the general formula.

This also means that we'll not be doing much work with the value of series in order to get the value we'll also need to know the general formula for the partial sums. We will continue with a few more example however, since this is technically how we determine convergence and the value of a series. Also, the remaining examples we'll be looking at in this section will lead us to a very important fact about the convergence of series. So, let's take a look at a couple more examples.

Example : Determine if the following series converges or diverges. If it converges determine its sum.

$$\sum_{n=2}^{\infty} = \frac{1}{n^2 - 1}$$

Solution : This is actually once of the few series in which we are able to determine a formula for the general term in the sequence of partial fractions. However, in this section we are more interested in the general idea of convergence and divergence and so we'll put off discussing the process for finding the formula until the next section. The general formula for the partial sums is,

$$s_n = \sum_{i=2}^{n} = \frac{1}{i^2 - 1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

and in this case we have,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{4}-\frac{1}{2n}-\frac{1}{2(n+1)}\right)=\frac{3}{4}$$

The sequence of partial sums converges and so the series converges also and its value is,

$$\sum_{i=2}^{n} = \frac{1}{n^2-1} = \frac{3}{4}$$

Example : Determine if the following series converges or diverges. If it converges determine its sum.

$$\sum_{n=0}^{\infty}(-1)^n$$

Solution : In this case we really don't need a general formula for the partial sums to determine the convergence of this series. Let's just write down the first few partial sums.

$$s_0 = 1$$
$$s_1 = 1-1=0$$
$$s_2 = 1-1+1=1$$
$$s_3 = 1-1+1-1=0 \text{ etc.}$$

So, it looks like the sequence of partial sums is,

$$\{s_n\}_{n=0}^{\infty} = \{1,0,1,0,1,0,1,......\}$$

and this sequence diverges since $\lim_{n\to\infty} s_n$ doesn't exist. Therefore, the series also diverges.

Example : Determine if the following series conveges or diverges. If it converges determine its sum.

$$\sum_{n=1}^{\infty}\frac{1}{3^n-1}$$

Solution : Here is the general formula for the partial sums for this series.

$$s_n = \sum_{i=1}^{n} = \frac{1}{3^{i-1}} = \frac{3}{4}\left(1-\frac{1}{3^n}\right)$$

Again, do not worry about knowing this formula. This is not something that you'll ever be asked to know in my class.

In this case the limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{3}{2}\left(1-\frac{1}{3^n}\right)=\frac{3}{2}$$

The sequence of partial sums is convergent and so the series will also be convergent. The value of the series is,

$$\sum_{i=1}^{\infty}\frac{1}{3^{n-1}} = \frac{3}{2}$$

As we already noted, do not get excited about determining the general formula for the sequence of partial sums. There is only going to be one type of series where you will need to determine this formula and the process in that case isn't too bad. In fact, you already know how to do most of the work in the process as you'll see in the next section.

So, we've determined the convergence of four series now. Two of the series converged and two diverged. Let's go back and examine the series terms for each of these. For each of the series let's take the limit as n goes to infinity of the series terms (not the partial sums!).

$\lim_{n\to\infty} n = \infty$ this series diverged

$\lim_{n\to\infty}\frac{1}{n^2-1} = 0$ this series converged

$\lim_{n\to\infty}(-1)^n$ doesn't exist this series diverged

$\lim_{n\to\infty}\frac{1}{3^2-1} = 0$ this series converged

Notice that for the two series that converged the series term itself was zero in the limit. This will always be true for convergent series and leads to the following theorem.

Theorem : If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof : First let's suppose that the series starts at n = 1. If it doesn't then we can modify things as appropriate below. Then the partial sums are,

$$s_{n-1}=\sum_{i=1}^{n-1}a_i=a_1+a_2+a_3+a_4+....+a_{n-1}$$

$$s_n=\sum_{i=1}^{n}a_i=a_1+a_2+a_3+a_4+....+a_{n-1}+a_n$$

Next, we can use these two partial sums to write,

$$a_n=s_n-s_{n-4}$$

Now because we know that $\sum a_n$ is convergent we also know that the sequence $\{s_n\}_{n=1}^{\infty}$ is also convergent and that $\lim_{n\to\infty} s_n=s$ for some finite value s. However, since $n-1\to\infty$ as $n\to\infty n\to\infty$ we also have $\lim_{n\to\infty} s_{n-1}=s$.

We now have,

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}(s_n-s_{n-1})=\lim_{n\to\infty} s_n-\lim_{n\to\infty} s_{n-1}=s-s=0$$

Be careful to not misuse this theorem! This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is NOT true. If $\lim_{n\to\infty} a_n=0$ the series may actually diverge! Consider the following two series.

$$\sum_{n=1}^{\infty}\frac{1}{n} \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}$$

In both cases the series terms are zero in the limit as n goes to infinity, yet only the second series converges. The first series diverges.

It will be a couple of sections before we can prove this, so at this point please believe this and know that you'll be able to prove the convergence of these two series in a couple of sections.

Again, as noted above, all this theorem does is give us a requirement for a series to converge. In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would violate the theorem.

Divergence Test

If $\lim_{n\to\infty} a_n\neq 0$ then $\sum a_n$ will diverge.

Again, do NOT misuse this test. This test only says that a series is guaranteed to diverge if the series terms don't go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge! Again, recall the following two series,

$$\sum_{n=1}^{\infty}\frac{1}{n} \quad \text{diverges}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \text{converges}$$

One of the more common mistakes that students make when the first get into series is to assume that if $\lim_{n\to\infty} a_n=0$ then $\sum a_n$ will converge. There is just no way to guarantee this so be careful!

Example : Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{4n^2-n^3}{10+2n^3}$$

Solution : With almost every series we'll be looking at in this chapter the first thing that we should do it take a look at the series terms and see if they go to zero of not. If it's clear that the terms don't go to zero use the Divergence Test and be done with the problem.

That's what we'll do here.

$$\sum_{n=1}^{\infty}\frac{4n^2-n^3}{10+2n^3}=-\frac{1}{2}\neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

The divergence test is the first test of many tests that we will be looking at over the course of the next several sections. You will need to keep track of all these tests, the conditions under which they can be used and their conclusions all in one place so you can quickly refer back to them as you need to.

Next we should talk briefly revisit arithmetic of series and convergence/divergence. As we saw in the previous section if $\sum a_n$ and $\sum b_n$ are both convergent series then so are $\sum ca_n$ and $\sum_{n=k}^{\infty}(a_n\pm b_n)$. Furthermore, these series will have the following sums or values.

$$\sum ca_n = c\sum a_n$$

$$\sum_{n=k}^{\infty}(a_n \pm b_n) = \sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$$

We'll see an example of this in the next section after we get a few more examples under our belt. At this point just remember that a sum of convergent sequences is convergent and multiplying a convergent sequence by a number will not change its convergence. We need to be a little careful with these facts when it comes to divergent series.

In the first case if $\sum a_n$ is divergent then $\sum ca_n$ will also be divergent (provided *c* isn't zero of course) since multiplying a series that is infinite in value or doesn't have a value by a finite value (i.e. *c*) won't change the fact that the series has an infinite or no value. However, it is possible to have both $\sum a_n$ and $\sum b_n$ be divergent series and yet have,

$$\sum_{n=k}^{\infty}(a_n \pm b_n)$$

be a convergent series. Now, since the main topic of this section is the convergence of a series we should mention a stronger type of convergence.

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ also converges. Absolute convergence is stronger than convergence in the sense that a series that is absolutely convergent will also be convergent, but a series that is convergent may or may not be absolutely convergent.

In fact if $\sum a_n$ converges and $\sum |a_n|$ diverges the series $\sum a_n$ is called conditionally convergent.

At this point we don't really have the tools at hand to properly investigate this topic in detail nor do we have the tools in hand to determine if a series is absolutely convergent or not. So we'll not say anything more about this subject for a while. When we finally have the tools in hand to discuss this topic in more detail we will revisit it. Until then don't worry about it. The idea is mentioned here only because we were already discussing convergence in this section and it ties into the last topic that we want to discuss in this section.

In the previous section after we'd introduced the idea of an infinite series we commented on the fact that we shouldn't think of an infinite series as an infinite sum despite the fact that the notation we use for infinite series seems to imply that it is an infinite sum. It's now time to briefly discuss this. First, we need to introduce the idea of a rearrangement.

A rearrangement of a series is exactly what is might sound like, it is the same series with the terms rearranged into a different order.

For example, consider the following the infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + ...$$

A rearrangement of this series is,

$$\sum_{n=1}^{\infty} a_n = a_2 + a_1 + a_3 + a_{14} + a_5 + a_9 + a_4 + ...$$

The issue we need to discuss here is that for some series each of these arrangements of terms can have a different values despite the fact that they are using exactly the same terms.

Here is an example of this. It can be shown that,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + ... = In\,2$$

Since this series converges we know that if we multiply it by a constant *c* its value will also be multiplied by *c*. So, let's multiply this by, $\frac{1}{2}$ to get,

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} + \frac{1}{16} ..... \frac{1}{2} = In\,2$$

Now, let's add in a zero between each term as follows.

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + 0 + \frac{1}{12} + 0 + ... = \frac{1}{2}\ In\,2$$

We know that if two series converge we can add them by adding term by term

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + ... = \frac{3}{2}\ In\,2$$

Facts :

Given the series $\sum a_n$,

1. If $\sum a_n$ is absolutely convergent and its value is s then any rearrangement of $\sum a_n$ will also have a value of s.
2. If $\sum a_n$ is conditionally convergent and r is any real number then there is a rearrangement of $\sum a_n$ whose value will be r.

Again, we do not have the tools in hand yet to determine if a series is absolutely convergent and so don't worry about this at this point. This is here just to make sure that you understand that we have to be very careful in thinking of an infinite series as an infinite sum.

There are times when we can (i.e. the series is absolutely convergent) and there are times when we can't (i.e. the series is conditionally convergent).

As a final note, the fact above tells us that the series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

must be conditionally convergent since two rearrangements gave two separate values of this series. Eventually it will be very simple to show that this series conditionally convergent.

Special Series

In this section we are going to take a brief look at three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however the main reason that we're going to look at two of them in this section is that they are the only types of series that we'll be looking at for which we will be able to get actual values for the series.

The third type is divergent and so won't have a value to worry about.

In general, determining the value of a series is very difficult and outside of these two kinds of series that we'll look at in this section we will not be determining the value of series in this chapter. So, let's get started.

Geometric Series

A geometric series is an series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^{n}$$

These are identical series and will have identical values, provided they converge of course.

If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) \\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r} \\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n \end{aligned}$$

Now, from Theorem from the Sequences section we know that the limit above will exist and be finite provided. $-1 < r \le 1$ However, note that we can't let $r = 1$ since this will give division by zero. Therefore, this will exist and be finite provided $-1 < r < 1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty} s^n = \frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

Note that in using this formula we'll need to make sure that we are in the correct form. In other words, if the series starts at $n = 0$ then the exponent on the r must be n. Likewise if the series starts at $n = 1$ then the exponent on the r must be $n - 1$.

Example : Determine if the following series converge or diverge. If they converge give the value of the series.

1. $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$ 2. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{3n}}$

Solution :

1. $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

This series doesn't really look like a geometric series. However, notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of a geometric series. We will just need to decide which form is the correct form. Since the series starts at $n = 1$ we will want the exponents on the numbers to be $n - 1$.

It will be fairly easy to get this into the correct form. Let's first rewrite things slightly. One of the n's in the exponent has a negative in front of it and that can't be there in the geometric form. So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n+2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n+2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n+1}4^{2}}{9^{n-2}9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 16(9)\frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and $r = \frac{4}{9} < 1$. Therefore, since $|r| < 1$ we know the series will converge and its value will be,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}}\frac{9}{5} = (144) = \frac{1296}{5}$$

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}}\frac{9}{5} = (144) = \frac{1296}{5}$$

1. $\sum_{n=1}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Again, this doesn't look like a geometric series, but it can be put into the correct form.

In this case the series starts at $n = 0$ so we'll need the exponents to be n on the terms. Note that this means we're going to need to rewrite the exponent on the numerator a little,

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{\left((-4)^{3}\right)^{n}}{5^{n}5^{n-1}} = \sum_{n=0}^{\infty} 5\frac{(-64)^{n}}{5^{n}} = \sum_{n=0}^{\infty} 5\frac{(-64)^{n}}{5^{n}}$$

So, we've got it into the correct form and we can see that $a = 5$ and $r = \frac{64}{5}$. Also note that $|r| \geq 1$ and so this series diverges.

Example : Use the results from the previous example to determine the value of the following series.

1. $\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$ 2. $\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$

Solution : 1. $\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1}$

In this case we could just acknowledge that this is a geometric series that starts at $n = 0$ and so we could put in into the correct form and be done with it.

However, this does provide us with a nice example of how to use the idea of stripping out terms to our advantage. Let's notice that if we strip out the first term from this series we arrive at,

$$\sum_{n=0}^{\infty} 9^{-n+2}4^{n+1} = 9^{2}4^{1} + \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 324 + \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$$

From the previous example we know the value of the new series that arises here and so the value of the series in this example is,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 324 + \frac{1296}{5} = \frac{2916}{5}$$

2. $\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$

In this case we can't trip out terms from the given series to arrive at the series used in the previous example.

However, we can start with the series used in the previous example and strip terms out of it to get the series in this example. So, let's do that. We will strip out the first two terms from the series we looked at in the previous example.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 9^1 4^1 + 9^0 4^3 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = 208 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$$

We can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{266}{5}$$

Notice that we didn't discuss the convergence of either of the series in the above example.

Here's why. Consider the following series written in two separate ways (i.e. we stripped out a couple of terms from it).

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$

Let's suppose that we know $\sum_{n=3}^{\infty} a_n$ is a convergent series. This means that it's got a finite value and adding three finite terms onto this will not change that fact. So the value of $\sum_{n=0}^{\infty} a_n$ is also finite and so is convergent.

Likewise, suppose that $\sum_{n=0}^{\infty} a_n$ is convergent. In this case if we subtract three finite values from this value we will remain finite and arrive at the value of $\sum_{n=3}^{\infty} a_n$. This is now a finite value and so this series will also be convergent. In other words, if we have two series and they differ only by the presence, or absence, of a finite number of finite terms they will either both be convergent or they will both be divergent.

The difference of a few terms one way or the other will not change the convergence of a series. This is an important idea and we will use it several times in the following sections to simplify some of the tests that we'll be looking at.

Telescoping Series

It's now time to look at the second of the three series in this section. In this portion we are going to look at a series that is called a telescoping series. The name in this case comes from what happens with the partial sums and is best shown in an example.

Example : Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}$$

Solution : We first need the partial sums for this series.

$$sn = \sum_{n=0}^{\infty} \frac{1}{i^2 + 3i + 2}$$

Now, let's notice that we can use partial fractions on the series term to get,

$$\frac{1}{i^2 + 3i + 2} = \frac{1}{(i+2)(i+1)} = \frac{1}{i+1} - \frac{1}{i+2}$$

I'll leave the details of the partial fractions to you. By now you should be fairly adopt at this since we spent a fair amount of time doing partial fractions back in the Integration Techniques chapter. If you need a refresher you should go back and review that section. So, what does this do for us? Well, let's start writing out the terms of the general partial sum for this series using the partial fraction form.

$$s_n = \sum_{i=0}^{n} \left(\frac{1}{i+1} - \frac{1}{i+2} \right)$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$+....+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=1-\frac{1}{n+2}$$

Notice that every term except the first and last term canceled out. This is the origin of the name telescoping series. This also means that we can determine the convergence of this series by taking the limit of the partial sums.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)=1$$

The sequence of partial sums is convergent and so the series is convergent and has a value of

$$\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}=1$$

In telescoping series be careful to not assume that successive terms will be the ones that cancel. Consider the following example.

Example : Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}$$

Solution : As with the last example we'll leave the partial fractions details to you to verify.

The partial sums are,

$$s_n=\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2}}{i+1}-\frac{\frac{1}{2}}{i+3}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\frac{1}{2}}{i+1}-\frac{\frac{1}{2}}{i+3}\right)$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+....+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right]$$

In this case instead of successive terms canceling a term will cancel with a term that is farther down the list. The end result this time is two initial and two final terms are left.

Notice as well that in order to help with the work a little we factored the out of the series.

The limit of the partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1}{2}\left(\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\right)=\frac{5}{12}$$

So, this series is convergent (because the partial sums form a convergent sequence) and its value is,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}=\frac{5}{12}$$

Note that it's not always obvious if a series is telescoping or not until you try to get the partial sums and then see if they are in fact telescoping. There is no test that will tell us that we've got a telescoping series right off the bat. Also note that just because you can do partial fractions on a series term does not mean that the series will be a telescoping series. The following series, for example, is not a telescoping series despite the fact that we can partial fraction the series terms.

$$\sum_{n=1}^{\infty}\frac{3+2n}{n^2+3n+2}=\sum_{n=1}^{\infty}\left(\frac{1}{n+2}+\frac{1}{n+2}\right)$$

In order for a series to be a telescoping we must get terms to cancel and all of these terms are positive and so none will cancel.

Next, we need to go back and address an issue that was first raised in the previous section. In that section we stated that the sum or difference of convergent series was also convergent and that the presence of a multiplicative constant would not affect the convergence of a series. Now that we have a few more series in hand let's work a quick example showing that.

Example : Determine the value of the following series.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^2+4n+3}-9^{-n+2}4^{n+1}\right)$$

Solution : To get the value of this series all we need to do is rewrite it and then use the previous results.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^2+4n+3}-9^{-n+2}4^{n+1}\right)$$

$$=\sum_{n=1}^{\infty}\frac{4}{n^2+4n+3}-\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}$$

$$=4\sum_{n=1}^{\infty}\frac{4}{n^2+4n+3}-\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}d$$

$$=4\left(\frac{5}{12}\right)-\frac{1296}{5}=\frac{3863}{15}$$

We didn't discuss the convergence of this series because it was the sum of two convergent series and that guaranteed that the original series would also be convergent.

Harmonic Series

This is the third and final series that we're going to look at in this paper. Here is the harmonic series.

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

The harmonic series is divergent and we'll need to wait until the next section to show that. This series is here because it's got a name and so I wanted to put it here with the other two named series that we looked at in this section. We're also going to use the harmonic series to illustrate a couple of ideas about divergent series that we've already discussed for convergent series. We'll do that with the following example.

Example : Show that each of the following series are divergent.

1. $\sum_{n=1}^{\infty}\frac{5}{n}$ 2. $\sum_{n=1}^{\infty}\frac{1}{n}$

Solution : 1. $\sum_{n=1}^{\infty}\frac{5}{n}$

To see that this series is divergent all we need to do is use the fact that we can factor a constant out of a series as follows,

$$\sum_{n=1}^{\infty}\frac{5}{n}=5\sum_{n=1}^{\infty}\frac{1}{n}$$

Now, $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent and so five times this will still not be a finite number and so the series has to be divergent. In other words, if we multiply a divergent series by a constant it will still be divergent.

2. $\sum_{n=1}^{\infty}\frac{1}{n}$

In this case we'll start with the harmonic series and strip out the first three terms.

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\sum_{n=1}^{\infty}\frac{1}{n}$$

$$\Rightarrow\sum_{n=4}^{\infty}\frac{1}{n}=\left(\sum_{n=4}^{\infty}\frac{1}{n}\right)-\frac{11}{6}$$

In this case we are subtracting a finite number from a divergent series. This subtraction will not change the divergence of the series. We will either have infinity minus a finite number, which is still infinity, or a series with no value minus a finite number, which will still have no value. Therefore, this series is divergent. Just like with convergent series, adding /subtracting a finite number from a divergent series is not going to change the fact the convergence of the series.

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