Fractional Variational Iteration Method for the Solution of Time-Fractional Coupled Non-liner Partial Differential Equations

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Abstract. Our main objective of this research is to study the time-fractional coupled nonlinear partial differential equations viz., the system of third order KdV equations and the generalized coupled Hirota Satsuma KdV system by using fractional variational iteration method (FVIM). Numerical results are obtained using Maple 13. Results obtained by FVIM reveal that this approach is very accurate, effective, simple to use.

Keywords: Fractional variational iteration method; Caputo fractional derivative; Non-linear phenomenon; Partial differential equation; Time-fractional coupled system

AMC Code: 35A15, 35C10, 35E15

1. Introduction

The advantage of applying fractional models of differential equations in physical systems is actually their non local property. Fractional order derivative is a non local property while integer order derivative is local in nature. It shows that the upcoming state of physical system is also dependent on all of its historical states in addition to its present state. Hence, the fractional models are more realistic and fractional derivatives are often used in mathematical modeling of acoustics, fluid mechanics, anomalous diffusion, electrochemistry, signal processing, biology, etc., [1–5]. Most of the nonlinear FDEs do not possess exact solutions, therefore some numerical techniques are necessary to be used. In the past two decades, the problem of handling numerical solutions of FDEs has attracted the attention of many researchers. They are taking keen interest in developing numerical techniques for FPDEs. Recently, various techniques have been developed to solve nonlinear FPDEs such as, Adomian decomposition method [6], differential transform method [7], homotopy perturbation method [8], homotopy analysis method [9], homotopy analysis transform method [10], homotopy perturbation transform method [11,12], etc.

The area of fractional calculus and fractional differential equations (FDEs) has many applications in applied sciences and technology. In fact, many physical processes can be modeled using differential equations, where the fractional order derivatives can be considered in comparison to the system of differential equations which involves the integer order derivatives. [13-16]. In particular, Sun et al. [17] introduced real world applications in many branches of science and engineering, where nonlocality plays a key role. In this study, they have marked the several researchers around the globe, who have successfully proposed and analyzed mathematical models involving differential equations involving fractional order derivatives.

Fernandez et al. [18] have considered an integral transform introduced by generalized multi-parameter Mittag-Leffler functions. Moreover, Prakash et al.[19] suggested a new computational technique; namely new iterative sumudu transform method to solve numerically nonlinear time-fractional Zakharov-Kuznetsov equation in two dimensions. Also, Yusuf et al [20] have investigated the time fractional dispersive long wave equation and its corresponding integer order.

Motivated by the above discussions, in this paper, we propose the fractional variational iteration method (FVIM) [21-23] to get the approximate analytic solution of nonlinear time-fractional coupled PDEs and results are compared with recently developed techniques homotopy perturbation transform method [24].

FVIM directly attacks the nonlinear FDEs without a need to find certain polynomials for nonlinear terms and gives result in an infinite series, that rapidly converges to analytical solution. This method does not require linearization, discretization, little perturbations or any restrictive assumptions.

The time –fractional coupled PDEs describe particle motion with memory in time. Space-fractional derivative arises when variations are heavy-tailed and describes particle motion that accounts for variation in the flow field over the entire system. Also, the fraction in the time derivative suggests a modulation or weighting of system memory. Therefore, the study of time-fractional coupled PDEs is very important.

This paper is organized in the following manner. Section 1 is introductory. In section 2, we represents the brief review of preliminary descriptions of Caputo fractional derivative. In section 3, the working of numerical method FVIM. Section 4, illustrates two numerical examples on which, FVIM is applied to find the approximate solutions. In last Section 5, we have concluded the paper.

2. Preliminaries

Definition 2.1 [3]. Consider a real function h (χ), χ > 0. It is called in space $C_\zeta, \zeta \in R$ if ∋ a real no. b (> ζ), s.t. h (χ) = $\chi^b h_1(\chi)$, $h_1 \in C[0,\infty)$. It is clear that $C_\zeta \subset C_\gamma$ if $\gamma \leq \zeta$.

Definition 2.2 [3]. Consider a function h (χ), χ > 0. It is called in space $C_\zeta^m, m \in N \cup \{0\}$ if $h^{(m)} \in C_\zeta$.

Definition 2.3 [3]. Left sided Caputo fractional derivative of h, $h \in C_{-1}^m, m \in N \cup \{0\}$,

$$D_t^\beta h(t) = \begin{cases} \left[I^{m-\beta} h^{(m)}(t)\right], m-1 < \beta < m, m \in N, \\ \frac{d^m}{dt^m} h(t),\ \beta = m, \end{cases}$$

a. $I_t^\zeta h(x,t) = \frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1} h(x,s)ds;\ \zeta, t > 0.$

b. $D_\tau^\nu V(x,\tau) = I_\tau^{m-\nu}\frac{\partial^m V(x,\tau)}{\partial t^m}, m-1 < \nu \leq m.$

c. $D_t^\zeta I_t^\zeta h(t) = h(t)\ , m-1 < \zeta \leq m, m \in N.$

d. $I_t^\zeta D_t^\zeta h(t) = h(t) - \sum_1^{m-1} h^k(0+)\frac{t^k}{k!}, m-1 < \zeta \leq m,, m \in N.$

e. $I^v t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(v+\zeta+1)} t^{v+\zeta}.$

Definition 2.4. Laplace transform of Caputo fractional derivative is

$$L[D^\alpha g(t)] = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} g^{(k)}(0), n-1 < \alpha \leq n.$$

3. Basic plan of FVIM

To illustrate the process of solution by FVIM, we consider the time-fractional coupled nonlinear partial differential equation

$$D_t^\alpha u(x,t) + S(x,t) + Q(x,t) = g(x,t), \qquad 0 < \alpha \leq 1, \tag{1}$$

By FVIM, correction functional is formed as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left[\frac{\partial^\alpha U_n(x,\tau)}{\partial t^\alpha} + S(x,t) + Q(x,t) - g(x,t)\right] (d\tau)^\alpha \tag{2}$$

where λ is a Lagrangian multiplier.

Now by variational theory, Lagrangian multiplier λ must satisfy

$\frac{\partial^\alpha \lambda}{\partial t^\alpha}|_{\tau=t}$ and $1 + \lambda|_{\tau=t} = 0$.

Consequently, we obtain λ = -1,

and hence, from Eq.(3), We have

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial^\alpha U_n(x,\tau)}{\partial t^\alpha} + S(x,t) + Q(x,t) - g(x,t)\right] (d\tau)^\alpha \tag{3}$$

Now from Eq. (3), we can obtain successive approximations $u_n(x, t)$, n ≥ 0. The functions u_n are restricted variations which means $\delta\widetilde{u_n} = 0$. In this way, we get sequences $u_{n+1}(x, t)$, n ≥ 0. Finally, the exact solution is obtained as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

4. Numerical Experiments

In this section, we apply the proposed technique FVIM to some test examples.

Example 4.1. Consider a linear time–fractional system of third order KdV equations

$$D_t^{\alpha}u = u_{xxx} + uu_x + vv_x,\ D_t^{\alpha}v = -2v_{xxx} + uv_x \quad 0 < \alpha \le 1, \tag{4}$$

with initial condition $u(x,0) = \left(3 - 6tanh^2\frac{x}{2}\right), v(x,0) = \left(3\sqrt{2}tanh^2\frac{x}{2}\right),$ (5)

when $\alpha = 1$, the exact solution of Eqs.(4) - (5) is

$u(x,t) = 3 - 6tanh^2\left(\frac{x+t}{2}\right),\ v(x,t) = 3\sqrt{2}tanh^2\left(\frac{x+t}{2}\right)$

The initial solution can be taken as $u_0 = \left(3 - 6tanh^2\frac{x}{2}\right), v_0 = \left(3\sqrt{2}tanh^2\frac{x}{2}\right)$, then

$$u_1(x,t) = u_0 - \int_0^t \left[\frac{\partial^\alpha u_0}{\partial\tau^\alpha} - \frac{\partial^3 u_0}{\partial x^3} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial v_0}{\partial x}\right](d\tau)^\alpha,$$

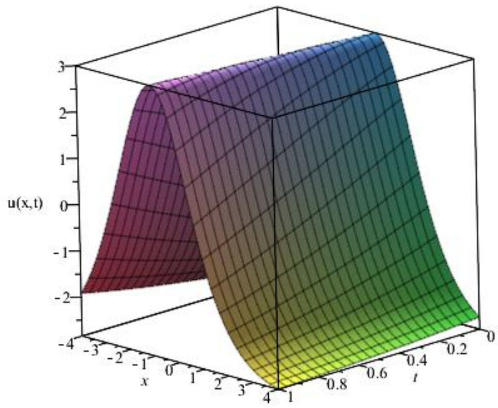
$$= \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\left(3 - \cosh x + \frac{t^\alpha \operatorname{sech}^3\left(\frac{x}{2}\right)\left(-19\sinh\left(\frac{x}{2}\right) + 5\sinh\left(\frac{3x}{2}\right)\right)}{\Gamma(1+\alpha)}\right),$$

$$v_1(x,t) = v_0 - \int_0^t \left[\frac{\partial^\alpha v_0}{\partial\tau^\alpha} + 2\frac{\partial^3 v_0}{\partial x^3} + u_0\frac{\partial v_0}{\partial x}\right](d\tau)^\alpha$$

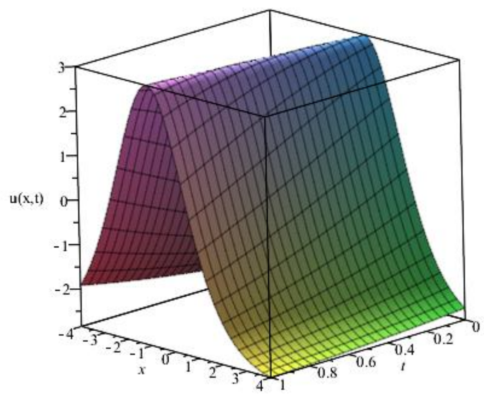
$$= 3\sqrt{2}\tanh^2\left(\frac{x}{2}\right) - \frac{t^\alpha}{\Gamma(1+\alpha)}\left(-3\sqrt{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)\left(3 - 6\tanh^2\left(\frac{x}{2}\right)\right) + 2\left(-6\sqrt{2}\operatorname{sech}^4\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) + 3\sqrt{2}\operatorname{sech}^2\left(\frac{x}{2}\right)\tanh^3\left(\frac{x}{2}\right)\right)\right)$$

Continuing in this way, the next iterations can be computed using Maple 13.
Finally, the solution is found as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),\ v(x,t) = \lim_{n\to\infty} v_n(x,t).$$



(a)



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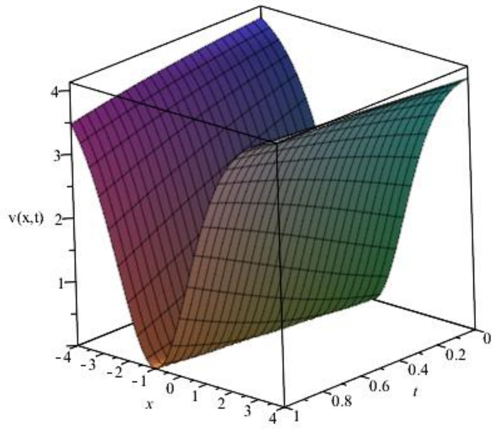

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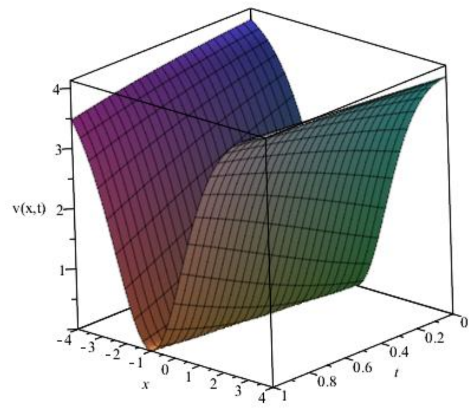


(d)

Fig. 1: Exact solution vs approximate solution. **(a)** Exact solution for u(x,t); **(b)** 3rd order approx. solution when $\alpha = 1$ for u(x,t); **(c)** Exact solution for v(x,t); **(d)** 3rd order approx. solution when $\alpha = 1$ for v(x,t);

Example 4.2 . Consider a linear time–fractional generalized KdV equations

$$D_t^{\alpha} u = \frac{1}{2} u_{xxx} - 3uu_x + 3(vw)_x,$$
$$D_t^{\alpha} v = 3uv_x - v_{xxx}, \qquad 0 < \alpha \leq 1, \tag{6}$$
$$D_t^{\alpha} w = 3uw_x - w_{xxx},$$

with initial condition

$$u(x,0) = \left(\frac{-1}{3} + 2tanh^3 x\right), v(x,0) = (\tanh x), \ w(x,0) = \left(\frac{8}{3}\tanh x\right) \tag{7}$$

when $\alpha = 1$, the exact solution of Eqs.(6) - (7) is

$$u(x,t) = \frac{-1}{3} + 2tanh^3(x+t), \ v(x,t) = \tanh(x+t), \ w(x,t) = \frac{8}{3}\tanh\ (x+t),$$

The initial solution can be taken as

$$u_0 = \left(\frac{-1}{3} + 2tanh^3 x\right), v_0 = (\tanh x), \ w_0 = \left(\frac{8}{3}\tanh x\right), \text{ then}$$

$$u_1(x,t) = u_0 - \int_0^t \left[\frac{\partial^{\alpha} u_0}{\partial \tau^{\alpha}} - \frac{1}{2}\frac{\partial^3 u_0}{\partial x^3} + 3\, u_0 \frac{\partial u_0}{\partial x} + 3 \frac{\partial v_0 w_0}{\partial x}\right] (d\tau)^{\alpha}$$
$$= \frac{-1}{3} + 2tanh^3 x - \frac{2t^{\alpha} sech^2 x\,(-3 + 2\tanh x\,(-4 + 12\,sech^2 x \tanh x - 3\,tanh^3 x + 9\,tanh^4 x))}{\Gamma(1+\alpha)},$$

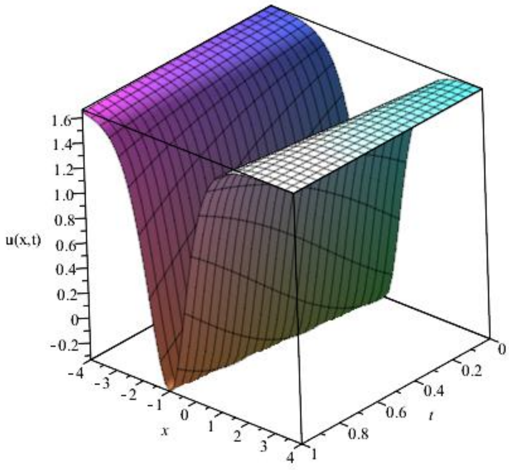
$$v_1(x,t) = v_0 - \int_0^t \left[\frac{\partial^{\alpha} v_0}{\partial \tau^{\alpha}} + \frac{\partial^3 v_0}{\partial x^3} + \ 3\, u_0 \frac{\partial v_0}{\partial x}\right] (d\tau)^{\alpha}$$
$$= \tanh x - \frac{t^{\alpha} sech^2 x\,(1 + 2sech^2 x - 2\,tanh^2 x\,(2 + 3\tanh x))}{\Gamma(1+\alpha)}$$

$$w_1(x,t) = w_0 - \int_0^t \left[\frac{\partial^{\alpha} v_0}{\partial \tau^{\alpha}} + \frac{\partial^3 w_0}{\partial x^3} + 3\, u_0 \frac{\partial w_0}{\partial x}\right] (d\tau)^{\alpha}$$

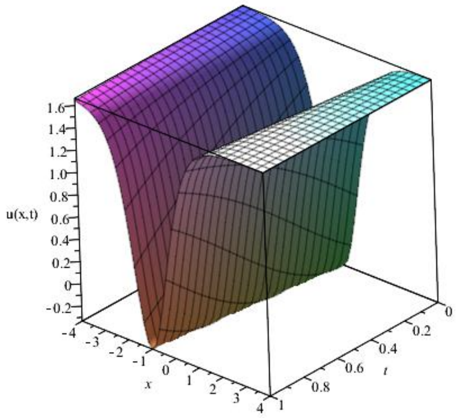
$$= \frac{8}{3}\left(\tanh x - \frac{t^{\alpha} sech^2 x\,(1+2sech^2 x - 4tanh^2 x + 6tanh^3 x\ )}{\Gamma(1+\alpha)}\right)$$

Continuing in this way, the next iterations can be computed using Maple 13.
Finally, the solution is found as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),\ v(x,t) = \lim_{n\to\infty} v_n(x,t)\ \&\ w(x,t) = \lim_{n\to\infty} w_n(x,t).$$



(a)



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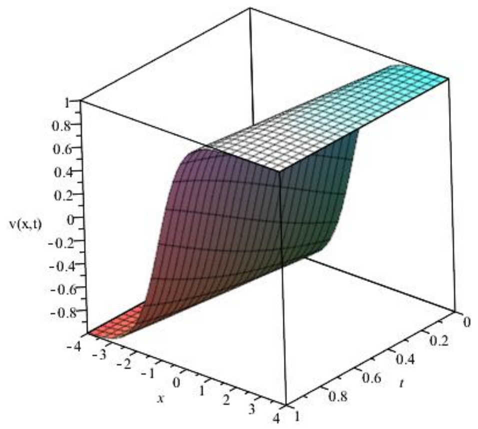
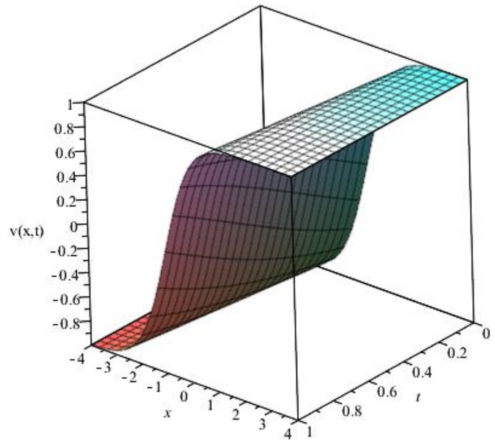
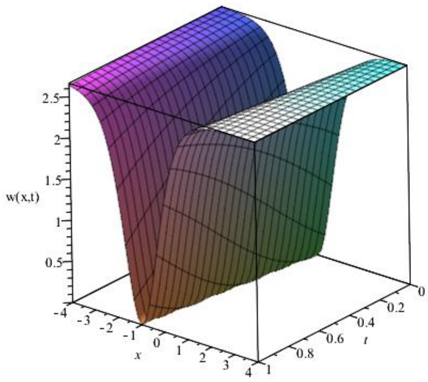

v(x,t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-4 -3 -2 -1 0 1 2 3 4
x
0 0.2 0.4 0.6 0.8 1
t

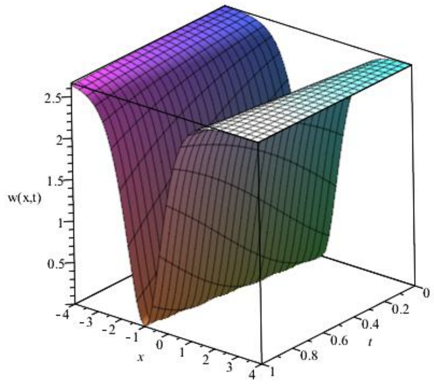
(c)


v(x,t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-4 -3 -2 -1 0 1 2 3 4
x
0 0.2 0.4 0.6 0.8 1
t

(d)


w(x,t)
2.5
2
1.5
1
0.5
-4 -3 -2 -1 0 1 2 3 4
x
0 0.2 0.4 0.6 0.8 1
t

(e)



(f)

Fig. 2: Exact solution vs approximate solution. **(a)** Exact solution for u(x,t); **(b)** 3rd order approx. solution when $\alpha = 1$ for u(x,t); **(c)** Exact solution for v(x,t); **(d)** 3rd order approx. solution when $\alpha = 1$ for v(x,t); **(e)** Exact solution for w(x,t); **(f)** 3rd order approx. solution when $\alpha = 1$ for w(x,t).

5. Conclusion

In this work, fractional variational iteration method (FVIM) has been successfully applied to obtain an approximate solution of coupled nonlinear partial differential equations. On comparing the results of this method with HPTM [12], it is observed that FVIM is extremely simple and easy to handle the nonlinear terms. Maple 13 package is used to calculate series obtained from iteration. Further, the method needs much less computational work, which shows the fast convergent for solving nonlinear partial differential equations.

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