The Commuting Toeplitz operators on the harmonic symbols

Osman Abdalla Adam Osman
Department of Mathematics – College of Science & Arts in n Uglat
Asugour– Qassim University – Saudi Arabia Email: o.osman@qu.edu.sa

Abstract : In Bergman's field, I show that the Toeblitz operators with its symbiotic symbols in some special cases .To manage and characterize the functions consistent with the invariant value property.

IndexTerms: Toeblitz , operator, Bergman, spaces, function, invariant, value

1- Introduction

Let dA denote the usual area measure on the open unit disk D in the complex plane C. The complex space $L^2(D, dA)$ is a Hilbert space with the inner product.

$$\langle f, g\rangle = \int_D f\bar{g}dA$$

The Bergman space L^2_a is the set of those functions in $L^2(D, dA)$ that are analytic on D. The Bergrnan space L^2_a is a closed subspace of $L^2(D, dA)$, and so there is an orthogonal projection P from $L^2(D, dA)$ onto L^2_a [1]. For $(\varphi + 1) \in L^\infty(D, dA)$, the Toeplitz operator with symbol $(\varphi + 1)$, denoted $T_{\varphi+1}$, is the operator from L^2_a to L^2_a defined by $T_{\varphi+1}f = P\{(\varphi + 1)f\}$. By a harmonic function we mean a complex-valued function on D whose Laplacian is identically 0.

Theorem 1: Suppose that $(\varphi + 1)$ and $(\psi + 1)$ are bounded harmonic functions on D. Then

$$T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$$

if and only if

(1.1) $(\varphi + 1)$ and $(\psi + 1)$ are both analytic on D,

(1.2) $\overline{(\varphi+1)}$ and $\overline{(\psi+1)}$ are both analytic on D,

(1.3) there exist constants $a, b \in C$, not both 0, such that $a(\varphi + 1) + b(\psi + 1)$ is constant on D.

We will see if the trend of that theory is trivial, but the proof of the "only if" direction requires a converse to an invariant form of the mean value property.

The clarification of Theory 1 is similar to the analogous result proved in [2] for Toeplitz operators with symbol in $L^\infty(\partial D)$ acting on the Hardy space $H^2(\partial D)$. Brown and Halmos proved their result by examining the matrix of products of Hardy space Toeplitz operators. On the Bergman space L^2_a, Toeplitz operators do not have nice matrices, and the techniques used by Brown and Halmos do not seem to work in this context. Thus function theory, rather than matrix manipulations, plays a large role in our proof.

A special case of Theorem 1 was proved in [3], using function theory techniques quite different from those that we use here. Also proved a special case of Theorem 1; our proof makes use of some of his ideas.

Functions in $L^\infty(\partial D)$ correspond, via the Poisson integral, to bounded harmonic functions on D, so the restriction in Theorem 1 to consideration only of Toeplitz operators with harmonic symbols is natural. More importantly, Theorem 1 does not hold if " We can replace measurable harmonic ". For example, Paul Bourdon has pointed out to us that if $(\varphi + 1)$ and $(\psi + 1)$ are any two radial functions in $L^\infty(D, dA)$, then $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$ (a function is called radial if its value at z depends only on $|z|$). Thus the following open problem may be hard: Find conditions on functions $(\varphi + 1)$ and $(\psi + 1)$ in $L^\infty(D, dA)$ that are necessary and sufficient for $T_{\varphi+1}$ to commute with $T_{\psi+1}$.

2- The constant property of the intermediate value

A continuous function on the disk D is hannonic if and only if it has the mean value property. We characterize hannonic functions in terms of an invariant mean value property.

Let $Aut(D)$ denote the set of analytic, one-to-one maps of D onto D (Where Aut stands for Automorphism). A function h on D is in $Aut(D)$ if and only if there exist $\alpha \in \partial D$ and $\beta \in D$ such that

$$h(z) = \alpha \frac{\beta - z}{1 - \bar{\beta} z}$$

for all $z \in D$

A function $u \in C(D)$ is said to have the invariant mean value property if

$$\int_0^{2\pi} u\left(h(re^{i\theta})\right) \frac{d\theta}{2\pi} = u(h(0)) \qquad (1)$$

for every $h \in Aut(D)$ and every $r \in [0,1)$. Here "invariant" refers to conformal invariance, meaning invariance under composition with elements of $Aut(D)$. If u is harmonic on D, then so is $u \circ h$ for every $h \in Aut(D)$; thus hannonic functions have the invariant mean value property. The converse is also true [4], if a function $u \in C(D)$ has the invariant mean value property, then u is harmonic on D.

The invariant mean value property concerns averages over circles with respect to arc length measure. Because we are dealing with the Bergman space L_a^2 , we need an invariant condition stated in terms of an area average over D. Thus we say that a function $u \in C(D) \cap L^1(D, dA)$ has the area version of the invariant mean value property if

$$\int_D u \circ h \frac{d\theta}{2\pi} = u(h(0)) \qquad (2)$$

for every $h \in Aut(D)$. If u is in $C(D) \cap L^1(D, dA)$, then so is $u \circ h$ for every $h \in Aut(D)$, so the left-hand side of the above equation makes sense. Note that the area version of the invariant mean value property deals with integrals over all of D, as opposed to integrals over rD for $r \in (0,1)$.

If u is harmonic on D and in $L^1(D, dA)$, then so is $u \circ h$ for every $h \in Aut(D)$. Thus, by the mean value property, harmonic functions have the area version of the invariant mean value property. Whether or not the converse is true is an open question. In other words, if $u \in C(D) \cap L^1(D, dA)$ has the area version of the invariant mean value property, must u be harmonic? This question has an affirmative answer if we replace the hypothesis that u is in $C(D) \cap L^1(D, dA)$ with the stronger hypothesis that u is in $C(\bar{D})$; see in [4,5].

We need to consider functions that are not necessarily continuous on the closed disk, so the result mentioned in the last sentence will not suffice. However, our functions do have the property that their radializations are continuous on the closed disk, and we will prove that this property, along with the area version of the invariant mean value property, is enough to imply harmonicity.

If $u \in C(D)$, then the radialization of u denoted $R(u)$, is the function on D defined by

$$R(u)(w) = \int_0^{2\pi} u(we^{i\theta}) \frac{d\theta}{2\pi} \qquad (3)$$

In the following lemma, which will be a key tool in our proof of Theorem 1, the statement $R(u \circ h) \in C(\bar{D})$ means $R(u \circ h)$ can be extended to a continuous complex valued function on $\bar{D}$

Lemma 2: Suppose that $u \in C(D) \cap L^1(D, dA)$. Then u is harmonic on D if and only if

$$\int_D u \circ h \frac{dA}{\pi} = u(h(0)) \qquad (4)$$

and

$$R(u \circ h) \in C(\bar{D}) \;\; for\ every\ h \in Aut(D) \qquad (5)$$

Proof : Suppose that u is harmonic on D. Let $h \in Aut(D)$. As we discussed earlier, $u \circ h$ is harmonic and so (4) holds. The mean value property implies that $R(u \circ h)$ is a constant function on D, with value $u(h(0))$, so (5) also holds.

To prove the other direction, suppose that (4) and (5) hold. Let $h \in Aut(D)$, and let

$$v \in R(u \circ h).$$

By (5), $v \in C(\bar{D})$.

We want to show that v has the area version of the invariant mean value property. To do this, fix $g \in Aut(D)$. Then

$$\int_D v \circ g \frac{dA}{\pi} u = \int_D R(u \circ h)(g(w)) \frac{dA(w)}{\pi} = \int_D \int_0^{2\pi} u\left(h(g(w)e^{i\theta})\right) \frac{d\theta}{2\pi} \frac{dA(w)}{\pi} \qquad (6)$$

To check that interchanging the order of integration in the last integral is valid, for each $\theta \in [0,2\pi]$ define $f_\theta \in Aut(D)$ by

$$f_\theta(w) = h(g(w)e^{i\theta})$$

The inverse f_θ^{-1} of f_θ is also an analytic automorphism of D, so there exist $\alpha \in \partial D$ and $\beta \in D$ such that

$$f_\theta^{-1}(z) = \alpha \frac{\beta - z}{1 - \bar{\beta} z} \quad \text{for all } z \in D$$

Thus

$$|(f_\theta^{-1})'(z)| = \frac{1 - |\beta|^2}{|1 - \bar{\beta} z|^2} \le \frac{1 + |\beta|}{1 - |\beta|} \text{for all } z \in D$$

Note that $\beta = f_\theta(0) = h(g(0)e^{i\theta})$; we are thinking of h and g as fixed, so the above inequality shows there is a constant K such that

$$|(f_\theta^{-1})'(z)| \le K \text{ for all } z \in D \text{ and all } \theta \in [0,2\pi]$$

Now

$$\int_0^{2\pi}\int_D |u(h(g(w)e^{i\theta}))| \frac{dA(w)}{\pi}\frac{d\theta}{2\pi} = \int_0^{2\pi}\int_D |u(z)||(f_\theta^{-1})'(z)|^2 \frac{dA(z)}{\pi}\frac{d\theta}{2\pi}$$

$$\leq K^2 \int_D |u(z)| \frac{dA(z)}{\pi} \leq \infty$$

That is apply Fubini's Theorem to (6), getting

$$\int_D v\circ g \frac{dA}{\pi} = \int_0^{2\pi}\int_D u(h(g(w)e^{i\theta})) \frac{dA(w)}{\pi}\frac{d\theta}{2\pi} = \int_0^{2\pi}\int_D (v\circ f_\theta)(w) \frac{dA(w)}{\pi}\frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} u(f_\theta(0))\frac{d\theta}{2\pi} = \int_0^{2\pi} u(h(g(0)e^{i\theta}))\frac{d\theta}{2\pi} = R(u\circ h)(g(0)) = v(g(0))$$

Thus v is a continuous function on $\bar{D}$ that has the area version of the invariant mean value property. Hence v is harmonic on D [4,5]. Because v is also a radial function, the mean value property implies that v is a constant function on D, with value $v(0)$.

Recall that $v = R(u\circ h)$, so

$$\int_0^{2\pi} (u\circ h)(re^{i\theta})\frac{d\theta}{2\pi} = u(h(0))$$

for every $r \in [0, 1)$ and for each $h \in Aut(D)$. In other words, u has the invariant mean value property. Thus in [4], u is harmonic on D.

As mentioner earlier, it is unknown whether Lemma 2 remains true if (5) is deleted. We believe that the following proposition, which reduces this question to a tempting integral equation, is the best way to attack this problem. Patrick Ahern and Walter Rudin also independently proved Lemma 2 and Proposition 3 at about the same time we did.

Proposition 3 : Suppose that the constant functions are the only functions $V \in C([0,1)) \cap L^{/}[0,1]$ such that

$$V(t) = (1-t)^2 \int_0^1 \frac{1+ts}{(1-ts)^3} V(s)ds \quad \text{for every } t \in [0,1) \qquad (7)$$

Then every function in $C(D) \cap L^{/}(D, dA)$ having the area version of the invariant mean value property is harmonic.

Proof: First suppose that v is a radial function in $C(D) \cap L^{/}(D, dA)$ having the area version of the invariant mean value property. We will show that v is constant on D. For $\alpha \in [0,1)$, let $h_\alpha \in Aut(D)$ be defined by

$$h_\alpha(z) = \frac{\alpha - z}{1-\alpha z}$$

Note that h_α is its own inverse under composition. For each $\alpha \in [0,1)$ we have

$$v(\alpha) = \int_D (v\circ h_\alpha)(z)\frac{dA(z)}{\pi} = \int_D v(w)|h_\alpha'(w)|^2\frac{dA(w)}{\pi}$$

$$= (1-\alpha^2)^2 \int_0^1 v(r)r \int_0^{2\pi} \frac{1}{|1-\alpha re^{i\theta}|^4}\frac{d\theta}{2\pi} dr$$

$$= (1-\alpha^2)^2 \int_0^1 \frac{r(1+\alpha^2 r^2)}{(1-\alpha^2 r^2)^3} v(r)dr$$

$$= (1-\alpha^2)^2 \int_0^1 \frac{1+\alpha^2 s}{(1-\alpha^2 s)^3} v(\sqrt{s})ds$$

In the above equation, replace α with $\sqrt{t}$ and define a function V on $[0, 1)$ by $V(t) = v(\sqrt{t})$, transforming the above equation into (7). Hence V is constant on $[0,1)$, and thus so is v, as claimed.

To complete the proof, now suppose that u is a function in $C(D) \cap L^{/}(D, dA)$ having the area version of the invariant mean value property. Let $h \in Aut(D)$. Clearly $R(u\circ h)$ is a radial function on D, and, as shown in the proof of Lemma 2, $R(u\circ h)$ has the area version of the invariant mean value property. By the above paragraph, $R(u\circ h)$ is constant on D. In particular, $R(u\circ h) \in C(\bar{D})$, and so by Lemma 2, u is harmonic .

3- The Toeplitz Operators

For $h \in Aut(D)$, define an operator U_h on L^2_a by

$$U_h f = (f\circ h)h'.$$

A simple computation shows that U_h is a unitary operator from L^2_a onto L^2_a, with inverse $U_{h^{-1}}$.

In the following lemma that proof of Theorem 1.

Lemma 4: Let $h \in Aut(D)$ and let $(\varphi + 1) \in L^\infty(D, dA)$. Then

$$U_h T_{\varphi+1} U_h^* = T_{(\varphi+1)\circ h}$$

Proof: Define $V_h: L^2(D,dA) \to L^2(D,dA)$ by $V_h f = (f^\circ h)h'$. Then V_h is a unitary operator from $L^2(D,dA)$ onto $L^2(D,dA)$. Obviously $V_h L_a^2 = U_h$. Thus V_h maps L_a^2 onto L_a^2, so

$$PV_h = V_h P \qquad (8)$$

If $f \in L_a^2$ then

$$T_{(\varphi+1)^\circ h} U_h f = T_{(\varphi+1)^\circ h}((f^\circ h)h') = P[((\varphi+1)^\circ h)(f^\circ h)h']$$
$$= P[V_h((\varphi+1)f)] = V_h[P((\varphi+1)f)] = U_h T_{\varphi+1} f \qquad \text{(from 8)}$$

Thus $T_{(\varphi+1)^\circ h} U_h = U_h T_{\varphi+1}$ and because U_h is unitary, this implies the desired result.

Let $H^p(D)$ denote the usual is the Hardy space on the disk. It is well known that $H^1(D) \subset L^2{}_a$ In the proof of Theorem 1 we will use, without comment, the following consequence: If $f, g \in H^2(D)$, then $fg \in L^2_a$, and thus $fg \in L^2(D,dA)$

We have now assembled all the ingredients needed to prove Theorem 1.

Proof of theorem 1: If we begin with the easy direction. First suppose that (1.1) holds, so that $(\varphi+1)$ and $(\psi+1)$ are analytic on D, which means that $T_{\varphi+1}$ and $T_{\psi+1}$ are, respectively, the operators on L_a^2 of multiplication by $(\varphi+1)$ and $(\psi+1)$. Thus $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$.

Now suppose that (1.2) holds, so that $\overline{(\varphi)}$ and $\overline{(\psi)}$ are analytic on D. By the paragraph above, $T_{\bar\varphi}T_{\bar\psi} = T_{\bar\psi}T_{\bar\varphi}$. Take the adjoint of both sides of this equation, and use the identity $T^*_{\bar\varphi} = T_{\varphi+1}$ to conclude that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$.

And finally suppose that (1.3) holds, so there exist constants $a, b \in C$, not both 0, such that $a(\varphi+1) + b(\psi+1)$ is constant on D. If $a \neq 0$, then there exist constants $\beta, \gamma \in C$ such that $(\varphi+1) = \beta(\psi+1) + \gamma$ on D, which means that $T_{\varphi+1} = \beta T_{\psi+1} + \gamma I$ (I denotes the identity operator on L^2_a), which clearly implies that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$. If $b \neq 0$, we conclude in a similar fashion that $T_{\varphi+1}$ and $T_{\psi+1}$ commute.

Now to prove the other direction of Theorem 1, suppose now that $(\varphi+1)$ and $(\psi+1)$ are bounded harmonic functions on D such that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$. Because $(\varphi+1)$ and $(\psi+1)$ are harmonic on D, there exist functions f_1, f_2, g_1 and g_2 analytic on D such that

$$(\varphi+1) = f_1 + \bar f_2 \text{ and } (\psi+1) = g_1 + \bar g_2 \text{ on } D \qquad (9)$$

Because $(\varphi+1)$ and $(\psi+1)$ are bounded on D, the functions f_1, f_2, g_1 and g_2 must be in $H^2(D)$.

Let 1 denote the constant function 1 on D. Then

$$T_{\varphi+1}T_{\psi+1}1 = T_{\varphi+1}(P(\psi+1)) = T_{\varphi+1}(P(g_1+\bar g_2)) = T_{\varphi+1}(g_1+\overline{g_2(0)})$$
$$= P([f_1+\bar f_2][g_1+\overline{g_2(0)}]) = f_1g_1 + \overline{g_2(0)}f_1 + P(\bar f_2 g_1) + \overline{f_2(0)}\,\overline{g_2(0)}$$

Thus

$$\langle T_{\varphi+1}T_{\psi+1}1,1\rangle = \langle f_1g_1 + \overline{g_2(0)}f_1 + \bar f_2 g_1 + \overline{f_2(0)}\,\overline{g_2(0)}, 1\rangle$$
$$= \int_D f_1g_1 + \overline{g_2(0)}f_1 + \bar f_2 g_1 + \overline{f_2(0)}\,\overline{g_2(0)}\, dA$$
$$= \pi[f_1(0)g_1(0) + f_1(0)\overline{g_2(0)} + \overline{f_2(0)}\,\overline{g_2(0)}] + \int_D \bar f_2 g_1 \, dA \qquad (10)$$

A sirnilar formula can be obtained for $\langle T_{\psi+1}T_{\varphi+1}1,1\rangle$.

Because $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$ we can set the right-hand side of (10) equal to the corresponding formula for $\langle T_{\psi+1}T_{\varphi+1}1,1\rangle$, getting

$$\int_D (\bar f_2 g_1 - f_1 \bar g_2)\frac{dA}{\pi} = \overline{f_2(0)}g_1(0) - f_1(0)\overline{g_2(0)} \qquad (11)$$

Let $h \in Aut(D)$. Multiplying both sides of the equation $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$ on the left and by U^*_h on the right, and recalling that U_h is unitary (so $U^*_h U_h = 1$), we get

$$U_h T_{\varphi+1} U^*_h U_h T_{\psi+1} U^*_h = U_h T_{\psi+1} U^*_h U_h T_{\varphi+1} U^*_h$$

Lemma 4 now shows that

$$T_{(\varphi+1)*h} T_{(\psi+1)*h} = T_{(\psi+1)*h} T_{(\varphi+1)*h} \qquad (12)$$

Composing both sides of the equations in (8) with h expresses each of the bounded harmonic functions $(\varphi+1)*h$ and $(\psi+1)*h$ as the sum of an analytic function and a conjugate analytic function:

$$(\varphi+1)*h = f_1 * h + \bar f_2 * h \text{ and } (\psi+1)*h = g_1 * h + \bar g_2 * h \text{ on } D \qquad (13)$$

Equation (11) was derived under the assumption that $T_{\varphi+1}T_{\psi+1} = T_{\psi+1}T_{\varphi+1}$;thus (12), combined with (14), says that (11) is still valid when we replace each function in it by its composition with h. In other words,

$$\int_D (\bar f_2 g_1 - f_1 \bar g_2) * h \frac{dA}{\pi} = \bar f_2(h(0))g_1(h(0)) - f_1(h(0))\bar g_2(h(0))$$

Letting

$$u = \bar f_2 g_1 - f_1 \bar g_2$$

the equation above becomes

$$\int_D u * h \frac{dA}{\pi} = u(h(0))$$

In the other words, u has the area version of the invariant mean value property.

We can want to show that u is harmonic on D. By the above equation and Lemma 2, we need only show that $R(u * h) \in C(\bar{D})$. To do this, represent the analytic functions $f_2 * h$ and $g_1 * h$ as Taylor series:

$$(f_2 * h)(z) = \sum_{n=0}^{\infty} \alpha_n z^n \quad and \quad (g_1 * h)(z) = \sum_{n=0}^{\infty} \beta_n z^n \quad for\ all\ z \in D$$

Because $(\varphi + 1) * h$ and $(\psi + 1) * h$ are bounded harmonic functions on D, (13) implies that $f_2 * h$ and $g_1 * h$ are in $H^2(D)$, so

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \quad and \quad \sum_{n=0}^{\infty} |\beta_n|^2 < \infty \qquad (14)$$

Now for $z \in D$ we have

$$R\left((\bar{f}_2 g_1) * h\right)(z) = \int_0^{2\pi} \overline{(f_2 * h)(ze^{i\theta})}(g_1 * h)(ze^{i\theta}) \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} \bar{\alpha}_n \beta_n |z|^{2n}$$

The inequalities in (14) imply that $\sum_{n=0}^{\infty} |\alpha_n \beta_n| < \infty$, so the above formula for $R\left((\bar{f}_2 g_1) * h\right)$ shows that $R\left((\bar{f}_2 g_1) * h\right) \in C(\bar{D})$; similarly, we get : $R((f_1 \bar{g}) * h) \in C(\bar{D})$. Thus $R(u * h) \in C(\bar{D})$, as desired. Thus at this stage of the proof we know that u is harmonic.

Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ denote the usual operators defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad and \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

If f is analytic, then the Cauchy-Riemann equations show that

$$\frac{\partial f}{\partial z} = f', \quad \frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0 \quad and \quad \frac{\partial \bar{f}}{\partial \bar{z}} = \bar{f'}$$

It is easy to check that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ obey the usual addition and multiplication formulas for derivatives and that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

Thus, because u is harmonic, we have

$$0 = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial u}{\partial z}\right) = 4\frac{\partial}{\partial \bar{z}}\left[\frac{\partial(\bar{f}_2 g_1 - f_1 \bar{g})}{\partial z}\right] = 4\frac{\partial}{\partial \bar{z}}(\bar{f}_2 g_1' - f_1' \bar{g}) = 4(\bar{f}_2' g_1' - f_1' \bar{g}')$$

Hence

$$f_1' \bar{g}' = \bar{f}_2' g_1' \qquad (15)$$

We finish the proof by showing that the above equation implies that (1.1), (1.2), or (1.3) holds. If g_1' is identically 0 on D, then (15) shows that either g_2' is identically 0 on D (so $(\psi + 1)$ would be constant on D and (1.3) would hold) or f_1' is identically 0 on D (so both $\overline{(\varphi+1)}$ and $\overline{(\psi+1)}$ would be analytic on D and (1.2) would hold). Similarly, if g_2' is identically 0 on D, then (15) shows that either (1.3) or (1.1) would hold. Thus we may assume that neither g_1' nor g_2' is identically 0 on D, and so (15) shows that

$$\frac{f_1'}{g_1'} = \overline{\left[\frac{f_2'}{g_2'}\right]}$$

at all points of D except the countable set consisting of the zeroes of $g_1' g_2'$. The left-hand side of the above equation is an analytic function (on D with the zeroes of $g_1' g_2'$ deleted), and the right hand side is the complex conjugate of an analytic function on the same domain, and so both sides must equal a constant $c \in C$. Thus $f_1' = c g_1'$ and $f_2' = \bar{c} g_2'$ on D. Hence $f_1 - c g_1$ and $\bar{f}_2 - \overline{c g_2}$ are constant on D, and so their sum, which equals $(\varphi + 1) - c(\psi + 1)$, is constant on D; in other words, (1.3) holds and the proof of Theorem 1 is complete .

Recall that an operator is called normal if it commutes with its adjoint. We can use Theorem 1 to prove the following corollary, which states that for $(\varphi + 1)$ a bounded harmonic function on D, the Toeplitz operator $T_{\varphi+1}$ is normal only in the obvious case.

Corollary 5: Suppose that $(\varphi + 1)$ is a bounded harmonic function on D. Then $T_{\varphi+1}$ is a normal operator if and only if $(\varphi + 1)(D)$ lies on some line in C.

Proof: First suppose that $(\varphi + 1)(D)$ lies on some line in C. Then there exist constants $\alpha, \beta \in C$, with $\alpha \neq 0$, such that $\alpha(\varphi + 1) + \beta$ is real valued on D. Thus $T_{\alpha(\varphi+1+\beta)}$ is a self-adjoint operator, and hence $T_{\varphi+1}$ which equals $\alpha^{-1}(T_{\alpha(\varphi+1)+\beta} - \beta I)$, is a normal operator.

To prove the other direction, suppose now that $T_{\varphi+1}$ is a normal operator. Thus $T_{\varphi+1}T_{\overline{\varphi+1}} = T_{\overline{\varphi+1}}T_{\varphi+1}$ and so Theorem 1 implies that $(\varphi + 1)$ and $\overline{(\varphi+1)}$ are both analytic on D (in which case $(\varphi + 1)$ is constant, so we are done) or there are constants $\alpha, \beta \in C$, not both 0, such that $a(\varphi + 1) + \overline{b(\varphi + 1)}$ is constant on D. The latter condition implies that $(\varphi + 1)(D)$ lies on a line .

REFERENCES

[1] Sheldon Axler* and Zeljko Cuckovic, Commuting Toeplitz Operator with Harmonic Symbols, Integral Equations and Operator Theory Vol. 14 (1991)

[2] Arlen Brown and P. R. Halmos , Algebraic properties of Toeplitz operators, J. Reine Angew. Math. 213 (1964), 89-102.

[3] Sheldon Axler and Pamela Gorkin , Algebras on the disk and doubly commuting multiplication operators, Trans. Amer. Math. Soc. 309 (1988),711-723.

[4] Walter Rudin, Function Theory on the Unit Bail of Cn, Springer-Verlag, New York 1980.

[5] J. Arazy, S. D. Fisher, and J. Peeter, Hankel operators on weighted Bergman spaces, American J. Math. 110 (1988), 989-1053