REVERSE VERTEX-MAGIC LABELINGS OF GRAPHS

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Abstract : A Reverse vertex-magic total labeling of a graph with v vertices and e edges is defined as a one-to-one map taking the vertices and edges onto the integers 1,2,…, $v+e$ with the property that the subtract of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex. Properties of these labelings are studied. It is shown how to construct labelings for several families of graphs, including cycles, paths, complete graphs of odd order and the complete bipartite graph $K_{n,n}$. It is also shown that labelings are impossible for some other classes of graphs.

IndexTerms – Vertex-magic Labeling, Regular Graphs, paths, Cycles.

I. INTRODUCTION

All graphs are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E(G)|$ and $v = |V(G)|$. A general reference for graph theoretic notions is [ ]. We will deal only with connected graphs, although the concepts apply equally to graphs with more than one connected component.

A labeling for a graph is a map that takes graph elements to numbers (usually positive or non-negative integers). In this paper the domain is the set of all vertices and edges_ while other labelings have used the vertex set alone, or more rarely the edge set alone. The most complete recent survey of graph labelings is [ ].

Various authors have introduced labelings that generalize the idea of a magic square. Kotzig and Rosa[ ], for example, defined a magic labeling to be a labeling on the vertices and edges in which the labels are the integers from 1 to $v+e$ and where the sum of labels on an edge and its two endpoints is constant. Related labelings have been studied by other authors and there are numerous variations in the terminology used. Readers are referred to[ ] for a discussion of these matters and a standardization of the terminology. That paper studies in detail the kind of labeling described above, which we will now refer to as an edge□magic total labeling.

In this paper we introduce the notion of a reverse vertex-magic labeling. This will be an assignment of the integers from 1 to $v+e$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex subtract to a fixed constant. More formally, a one-to-one map f from $E \cup V$ onto the integers $\{1,2,...,e+v\}$ is a reverse vertex-magic labeling if there is a constant k so that for every vertex x

$$f(x) - \sum f(x,y) = k \qquad \rightarrow (1)$$

where the sum is over all vertices y adjacent to x. Let us call the sum of labels at vertex x the weight of the vertex, so we require $wt(x) = k$ for all x. The constant k is called the magic constant for f .

It is not hard to find examples of labelings for some graphs. One labeling for the graph $K_4 - e$ is shown in Figure 1. On the other hand, not every graph has a labeling. For the graph K_2,since $f(x) \neq f(y)$, then $f(x) - f(x,y) \neq f(y) - f(x,y)$, and so no labeling is possible.

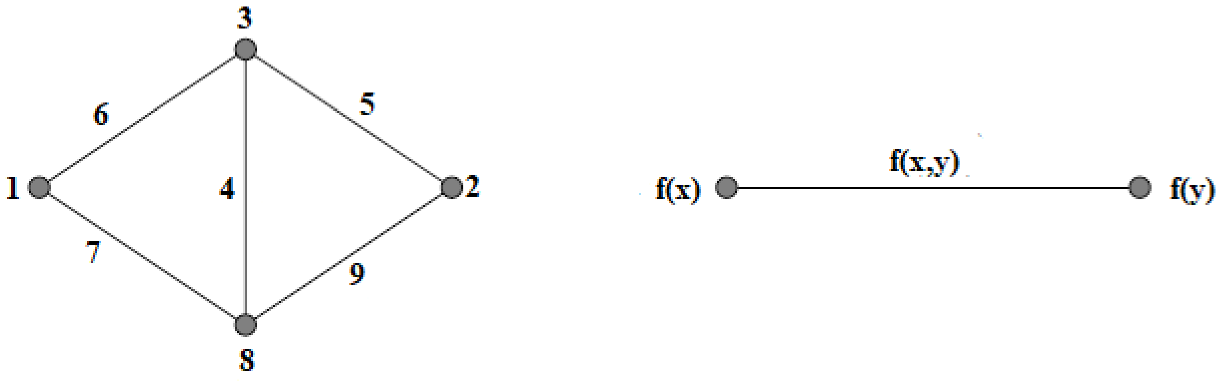

Figure 1: $K_4 - e$ has a labeling, P_2 does not.

In subsequent sections_ we show how to construct reverse vertex-magic labelings for several families of graphs and in some cases carry out complete enumerations of the admissible labelings. In addition, we determine several classes of graphs which cannot be labeled. The general flavour of the results that we have discovered so far can be summed up very roughly as follows: if there is not much variation among the degrees of the vertices then the graph possesses a labeling; if there is, then it does not.

2. Basic Counting

Set $M = e + v$ and let S_v be the sum of the vertex labels and S_e the sum of the edge labels. Clearly, since the labels are the numbers 1,2,…,M, we have as the sum of all labels

$$S_v + S_e = \sum_{1}^{M} i = (M+1)C_2.$$

At each vertex x_i we have $f(x_i) - \sum f(x, y) = k$. We sum this over all v vertices x_i. This subtracts each vertex label once and each edge label twice, so that

$$S_v - 2S_e = vk \qquad \rightarrow (2)$$

Combining these two equations gives us

$$(M+1)C_2 - 3S_e = vk \qquad \rightarrow (3)$$

The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the e smallest labels or, at the other extreme, the e largest labels, or anything between. Consequently we have

$$\sum_{1}^{e} i \le S_e \le \sum_{e+1}^{M} i \qquad \rightarrow (4)$$

A similar result holds for S_v. Combining (3) and (4) we get

$$(M+1)C_2 - 3\{(e+1)C_2\} \le vk \le 3\{(v+1)C_2\} - 2\{(M+1)C_2\}$$

which will give the range of feasible values for k.

It is clear from (1) that when k is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. Surprisingly, however, the vertex labels do not completely determine the labeling. Having assigned the vertex labels to a graph, it may be possible to assign the edge labels to the graph in several different ways. Figure 2 shows two labelings of W_4, which have the same vertex labels but a different edge labeling.

3. Regular Graphs

If a regular graph possesses a labeling, we can create a new labeling from it. Given a labeling f for any graph G, define the map f' on $E \cup V$ by

$f'(x_i) = M + 1 - f(x_i)$ for any vertex x_i, and

$f'(xy) = M + 1 - f(xy)$, *for any edge xy*

Clearly f' is also a one-to-one map from the set $E \cup V$ to $\{1, 2, ..., e+v\}$ and we will call f' the dual of f. We have the following:

Theorem 1: The dual of a reverse vertex-magic labeling for a graph G is a labeling if and only if G is regular.

Proof:- Suppose f is a reverse vertex-magic labeling for G with $wt(x_i) = k$. Then

$$\begin{aligned} wt'(x_i) &= f'(x_i) - \sum f'(xy) \\ &= M + 1 - f(x) - \sum (M + 1 - f(xy)) \\ &= (r+1)(M+1) - k. \end{aligned}$$

where r is the number of edges incident at x. Clearly this is constant if and only if r is constant; in other words, if and only if G is regular. Then $(r+1)(M+1)-k$ is the magic sum for the dual.

The general problem of whether one can use a reverse vertex-magic labeling of a graph G to produce a labeling of some sub graph or super graph of G appears to be very difficult. The next theorem answers a very special case of this question for regular graphs.

Theorem 2: Let G be a regular graph having a reverse vertex-magic labeling in which the label 1 is assigned to some edge e'. Then the graph $G - e'$ has a reverse vertex-magic labeling.

Proof:- Suppose G is an r-regular graph. Define a new mapping g by $g(x_i) = f(x_i) - 1$ and $g(xy) = f(xy) - 1$. Now g is a one-to-one mapping from $E \cup V$ to $\{0, 1, 2, ..., e+v-1\}$. If we now delete the edge e' which is labeled 0 by g) from G, then g is a one-to-one map from $(E - e') \cup V$ to $\{0, 1, 2, ..., e+v-1\}$. Also

$$\begin{aligned} wt_g(x_i) &= f(x_i) - 1 - \sum (f(xy) - 1) \\ &= wt_f(x_i) - r - 1 \\ &= k - r - 1. \end{aligned}$$

Thus $wt_g(x_i)$ is a constant k' and so g is a reverse vertex-magic labeling for $g - e'$.

Theorem 3:- If G is a regular graph and e an edge such that G - e has a reverse vertex-magic labeling, then that labeling is derived from a reverse vertex-magic labeling of G by the process described in Theorem 2.

Proof:- Suppose G is an r-regular graph and let f be any reverse vertex-magic labeling for G - e where e is the edge x_1x_2. Adjoin the edge x_1x_2 to G - e and define f(x_1x_2). Now simply adding 1 to all the labels defines a new mapping f' which is easily seen to be a reverse vertex-magic labeling for G having the label 1 appearing on the edge x_1x_2. If the reverse vertex-magic of f is k, then the reverse vertex-magic of f' is $k + r + 1$.

4.Cycles and Paths

The easiest regular graphs to deal with are the cycles. For cycles (and only for cycles) a opposite vertex-magic labeling is equivalent to a reverse edge-magic labeling and the reverse edge-magic labelings have already received some attention(see[ ] and the other papers cited there).

Theorem 4:- The n-cycle C_n has a reverse vertex-magic labeling for any $n \geq 3$.

Proof:- The constructions of Theorems of [ ] provide a reverse edge-magic labeling g for C_n for every $n \geq 3$. In other words there is a constant k so that

$$f'(x_i x_{i+1}) - [f'(x_i) + f'(x_{i+1})] = k$$

for all vertices x_i of C_n.

If we define a new mapping f by $f(x_i) = f'(x_i x_{i+1})$ and $f(x_i x_{i+1}) = f'(x_{i+1})$, then we clearly have k as the weight at each vertex and so f is a reverse vertex-magic labeling of G.

Corollary 1:- The path P_n with n vertices has a reverse vertex – magic labeling for any $n \geq 3$.

Proof:- For each $n \geq 3$ at least one of the reverse edge-magic labelings f' referred to in the proof of the theorem assigns the label 1 to a vertex. Then the corresponding reverse vertex-magic labeling f will assign the label 1 to some edge e. By Theorem 2, C_n- e will have a reverse vertex-magic labeling.

Corollary 2:- Every labeling of P_n is derived from a labeling of C_n.

Proof:- This follows immediately from Theorem 3.
Using equations (3) and (4), we can readily determine the feasible values of k for the n-cycle. We find

$$\frac{-5v-1}{2} \leq k \leq \frac{v-1}{2}$$

A systematic search has found exactly 4 labelings for the 3-cycle; one for each feasible value of k. As mentioned above, once k is given, the edge labels completely determine the labeling, so we list only the edge labels.

k = 1	1,2,3
k = -2	1,3,5
k =-5	2,4,6
k = -8	4,5,6

Since the cycles are regular graphs, the duality described in Section 2 applies. The labeling with k= -8 is dual to the labeling with k = 1, and the labeling with k = -5 is dual to that with k= -2.

There are exactly 6 labelings for the 4-cycle and again they come in dual pairs. Every feasible value of k admits a labeling. They are listed below, with the edge labels given in cyclic order:

k = 0	1,3,2,6
k = -3	1,4,6,5
k = -3	1,5,2,8
k = -6	3,4,8,5
k = -6	1,7,4,8
k = -9	3,7,6,8

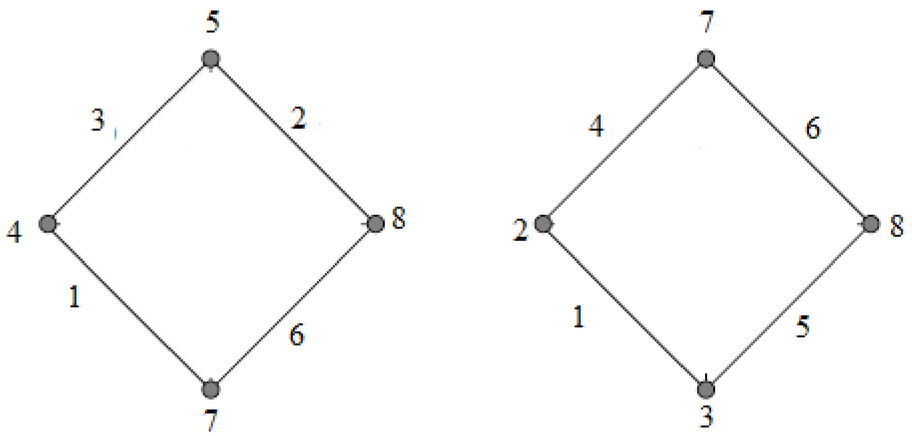

Figure 2: Two labelings of 4-Cycles

For the 5-cycle, we find $-13 \leq k \leq 2$. There are 6 labelings. For k=2 there is a unique solution (1,4,2,5,3) and for k = -2 we find (1,5,9,3,7) and also (1,7,3,4,10). Each of these has a dual.

For C_6, C_7, C_8, C_9, C_{10}, we identify the number of non-isomorphic labelings to be 20,118, 282,7092 respectively.

One of the Olympic rings problem referred to in the Introduction can be interpreted as asking for a labeling on the path of 5 vertices; one of the solutions the students found is shown in Figure 3.

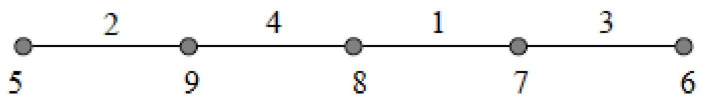

Figure 3: One labeling of P_5.

In the case of a path with v vertices, we have $e = v - 1$ and $M = v + e = 2v - 1$ and therefore

$$v(2v-1) - 3S_e = vk$$

So $S_e \equiv 0 (\bmod v)$ and consequently, from (2) also. Equations (3) and (4) then give us

$$\frac{1-v}{2} \leq k \leq \frac{v+1}{2} \qquad \rightarrow (5)$$

For a path with 3 vertices, equation (5) implies that $-1 \leq k \leq 2$. According to Theorem (5), the labelings of P_3 will be derived from those of C_3 which assign the label 1 to an edge. There are two such listed previously and they produce the labelings for P_3 shown in Figure 4.

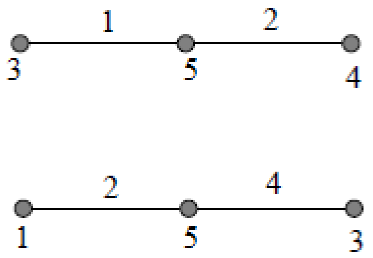

Figure 4: The two labelings of the path P_3.

For the path with 4 vertices, equation (5) implies that $-5 \leq k \leq 1$. Four of the six labelings of the 4-cycle listed previously yield a derived labeling for the 4-path. This time there is a labeling for all feasible values of k (we list the edge labels only):

k = 1	2,1,5
k = -2	4,5,3
k = -2	4,1,7
k = -5	6,3,7

For v = 5 , equation (5) implies that $-3 \leq k \leq 3$.

k =3	2,4,1,3
k = -3	4,8,2,6
k= -3	9,3,2,6

Further systematic search discovered 10 labelings for $v = 6$, there are 66 for $v = 7$ and for $v = 8$ there are 131 labelings.

REFERENCES

[1] W.S.Andrews, Magic Squares and Cubes Dover, 1960.
[2] J.A. Gallian, A Dynamic survey of graph labeling, Electronic. J.Combinatorics, 5 (1998), #DS6.
[3] A.Kotzig and A.Rosa, Magic valuations of finite graphs, Canad.Math. Bull.13(1970), 451-460.
[4] F.A. Muntaner-Batle, Magic graphs, **PhD** Thesis, Universita Politecnica de Catalunya, Spain (2001).
[5] F.A. Muntaner-Batle, Path-like tree, personal communication.
[6] J.A MacDougall and W. Wallis, Strong edge magic graphs of maximum size.
[7] S. ShariefBasha , K. Madhusudhan Reddy & MD.Shakeel, Algorithm To Construct Reverse Super Vertex Magic Labeling Of Complete
Graphs, Global Journal of Pure and Applied Mathematics, Volume 11, Number 1 (2015), pp. 1-7.
[8] Slamin, M. Baca, Y. Lin, M. Miller and R. Simanjuntak, Edge-magic totallabelings of wheels, fans and friendship graphs.
[9] S.VenkataRamana ,S.Shareefhbasha, Reverse super vertex-magic labeling of a graph, Ph.D. Thesis.