The Simplest realistic model of interaction potentials in study of Intertwining operators

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Abstract

A suitable model of interaction potentials can be used in detailed study of intertwining operators in a quantum mechanics. The study of intertwining operators used in atomic physics, molecular physics, nuclear physics and particle physics. The formalism and the techniques of SUSY quantum mechanics is generalized to the cases where the super potential is generated/ defined by higher excited eigenstates. The SUSY formalism applies everywhere between the singularities. A systematic application of the formalism to the other potentials with known spectra would yield an infinitely rich class of "solvable" potentials in terms of their partner potentials.

Key words: Intertwining operator, ordinary oscillator and discretely spiked oscillator.

1. Introduction:

Now a days, study of interesting feature of quantum field theory using reduction of space time to a point takes place [4]. In its turn, the quantization recipe becomes non unique. Such a type of ambiguity was unexpected. A certain complexified version of quantum mechanics [7] obtained using one dimensional space time continuum. In such a slightly more realistic setting the ambiguity results from the indeterminate complex asymptotic boundary conditions [3, 12].

Within this fresh methodical framework, there is detailed study of traditional concept of quantum mechanics [1,7]. In an attempt to apply the PT symmetric formalism of the quantum mechanics of more particles [3,8,9], we were forced to return to the intertwining operators. This provided a key motivation for our forthcoming discussion. The PT symmetric approach of the field theory opens new unresolved questions. The Witten's super symmetry in quantum mechanics [1, 5, 6] did also find its natural PT symmetrized new versions [12]. An explicit demonstration of their applicability to the intertwining operators is still missing and will in fast be a core of our present work.

2. INTERTWINING OPERATORS:

A pair of state generation operators which affecting the orbital angular momentum quantum number. 'l' and correspond to diagonal traversal of table associated with states and energies of discretely spiked harmonic oscillators. Such a pair of operators are said to be intertwining operators.

Let us define two operators b_l *and its adjoint* b_l^+ *as*

$$b_l = \frac{d}{dx} + \beta_l(x) \qquad \text{….(1)}$$

and $$b_l^+ = -\frac{d}{dx} + \beta_l(x) \qquad \text{…..(2)}$$

where $$\beta_l(x) = \left(x + \frac{l}{x}\right) \qquad \text{……(3)}$$

Function $\beta_l(x)$ and operator $\frac{d}{dx}$ do not commute i.e.

$$[\,\beta_l, \frac{d}{dx}] = -\beta_l'(x) = \left(\frac{l}{x^2} - 1\right) \qquad \text{…….(4)}$$

and as a consequence, the commutator $[b_l, b_l^+]$ is not zero either, since

$$b_l^+ b_l = -\frac{d^2}{dx^2} + \beta_l^2(x) - \beta_l'(x) = H_l + (2l - 1) \qquad \text{..….(5)}$$

and $$b_l b_l^+ = -\frac{d^2}{dx^2} + \beta_l^2(x) + \beta_l{}'(x) = H_{l-1} + (2l + 1) \qquad \text{…….(6)}$$

where $$H_{-1} = H_0 = -\frac{d^2}{dx^2} + x^2 \qquad \text{…….(7)}$$

and where the remaining Hamiltonians are defined by

$$H_l = -\frac{d^2}{dx^2} + x^2 + \frac{l(l+1)}{x^2} \qquad \text{……(8)}$$

Where $l = 0,1,2,3, \dots \dots$

With the help of above equations, we can establish two very useful formulas. Now we have

$$H_l b_l^+ - b_l^+ H_{l-1} = 2b_l^+ \qquad \text{…..(9)}$$

and $$H_{l-1} b_l - b_l H_l = -2b_l \qquad \text{….(10)}$$

These are quasi-commutative relations, connecting two "neighbouring" Hamiltonians. Let us now consider specific case of $\beta_l(x)$.

[A] For Ordinary oscillators: The case $l = 0$ is very special since

$$b_0^+ b_0 = H_0 - 1 \qquad \text{……(11)}$$

and $\quad b_0 b_0^+ = H_{-1} + 1 = H_0 + 1 \qquad (\because\ H_{-1} = H_0)$

Both equations refer to the same Hamiltonian of ordinary harmonic oscillators and the operators b_0 *and* b_0^+ degenerate to the familiar ladder operators

$$a_0^+ = b_0^+ = -\frac{d}{dx} + x \qquad \text{…….(12)}$$

and $a_o = b_o = \frac{d}{dx} + x$ …..…(13)

which have these important properties:

$$[a_0, a_o^+] = 2 \qquad \text{………(14)}$$

$$[H_0, a_o] = -2a_o \qquad \text{………(15)}$$

and $\quad [H_0, a_o^+] = 2a_o^+$ ………(16)

Consequently, the energies are given by

$E_{o,k} = (2k + 1)$: for $k = 0,1,2, \dots$, in units of $\frac{1}{2}\hbar\omega$. We will skip other details, since this is the very known case.

[B] <u>For Discretely spiked oscillators</u>:

First, let us prove that $b_{l+1}^+ |l, k>$ is an eigenvector of Hamiltonian H_{l+1} by assuming that this eigenequation holds

$$H_l \big| l, k >= E_{l,k} \big| l, k > \qquad \text{…….(17)}$$

and making use of formula (9) we get

$$H_{l+1} | b_{l+1}^+ (l, k) >= H_{l+1} b_{l+1}^+ | l, k >$$

$$= b_{l+1}^+ H_l | l, k > + 2 b_{l+1}^+ | l, k >$$

$$H_{l+1} \big| b_{l+1}^+ (l, k) >= (E_{l,k} + 2) \big| b_{l+1}^+ l, k > \qquad \text{…….(18)}$$

The operator b_{l+1}^+ transforms the state |l, k> with energy $E_{l,k}$into the state $|l + 1, k >$ with energy $E_{l+1,k} = (E_{l,k} + 2)$

such that

$$|l + 1, k >= \frac{1}{\sqrt{2k+4l+4}} b^+{}_{l+1} | l, k > \qquad \text{……(19)}$$

This corresponds to the diagonal move in NE direction in the table for states and energies of spiked harmonic oscillator. One can verify the normalization factor in equation

(19) by comparing the norm of its both sides, by making use of equation (6) and noticing that the energy of l^{th} oscillator in the state $|l, k >$ is given by

$$E_{l,k} = 2(l + k) + 1 \qquad \text{.....(20)}$$

Similarly, we can prove that state $b_{l+1}|l + 1, k >$ is the eigenvector of Hamiltonian H_l, given by

$$H_l(b_{l+1}|l + 1, k >) = (b_{l+1}H_{l+1} - 2b_{l+1}|l + 1, k >)$$
$$= (E_{l+1,k} - 2b_{l+1}|l + 1, k >) \qquad \text{......(21)}$$

The operators b_{l+1} transforms the state $|l + 1, k >$ with energy $E_{l+1,k}$ into the state $|l, k >$ with energy $E_{l,k} = (E_{l+1,k} - 2)$ such that

$$|l, k > = \frac{1}{\sqrt{2k+4l+4}} b_{l+1}|l + 1, k > \qquad \text{........(22)}$$

This corresponds to the diagonal move in SW direction in the table for states and energies of spiked harmonic oscillators as tabulated below.

Table :States and energies of discretely spiked harmonic oscillators

$l = 0$	$l = 1$	$l = 2$	$l = \cdots$ ...	Energy
....	....	....	....	....
....	....	$\|2, k >$	....	$(2k + 5)$
....	$\|1, k >$	$\|2, k - 1 >$	....	$(2k + 3)$
$\|0, k >$	$\|1, k - 1 >$	$\|2, k - 2 >$	....	$(2k + 1)$
$\|0, k - 1 >$	$\|1, k - 2 >$	....	....	$(2k - 1)$
$\|0, k - 2 >$	… … … … ..	....	....	$(2k - 3)$
....	....	....	....	....
....	....	$\|2,2 >$	....	9
....	$\|1,2 >$	$\|2,1 >$	....	7
$0,2 >$	$\|1,1 >$	$\|2,0 >$	....	5
$\|0,1 >$	$\|1,0 >$	....	....	3
$\|0,0 >$	....	....	....	1

Thus, it is clear that Intertwining operators $b_l \; and \; b_l^+$ can be used in study of states and energies of discretely spiked harmonic oscillators.

3. Results and Discussion:

In this research paper, I shall try to implant a certain more satisfactory mathematical symmetry into the set of harmonic oscillator wave functions. What can we expect from moving to larger semintegers, $\propto = |\delta|$? Just a strengthening of the tendencies which were revealed in Table 1 and 2. There features can be simply extrapolated. Thus, a $\delta = \frac{3}{2}$ modification of table 1 will contain two more lines at its bottom. At the next positive, $\delta = \frac{5}{2}$ supersymmetry between $H_L = H^{5/2} - 7$ and $H_R = H^{\frac{7}{2}} - 5$ will introduce a ground state mapping i.e. $L_0^{-\frac{5}{2}} \xrightarrow{A^{\frac{5}{2}}} L_0^{-\frac{-7}{2}}$. It appears at $E_{\frac{L}{R}} = -10$, witnessing just a continuing downward shift of the levels with the negative and decreasing superscripts.

Upto a constant shift, the PT supersymmetric partners coincide is given by the solution of the algebraic equation $|\delta| = |\delta + 1|$. This solution is unique $\left(for \; \delta = -\frac{1}{2}\right)$ and corresponds to the case where the poles in $A^\delta \; and \; B^\delta$ vanish. This is the only case tractable also without the use of the PT regularization.

A very exceptional role is played by the integers limits of $\propto$. In contrast, no special attention must be paid to the limit of vanishing spike $\left(\propto \to \frac{1}{2}\right)$. In general, with $\propto \neq \frac{1}{2}$, our PT regularization ($\in \neq 0$) can also be removed, if needed, via the limiting transition ($\in \to 0$) accompanied by the necessary halving of the axis of co-ordinates. This means that we have to replace $r(x) = (x - i \in)$ by the radial and real $r \in (0, \infty)$ and cross out all the states with $\beta < 0$. They are simply proclaimed "in acceptable" in the light their conventional interpretation. Intertwining operators provides a powerful techniques to explain the states and energies of discretely spiked harmonic oscillators.

4. Conclusion:

The formalism enables us to construct the creation and annihilation operators. We show, how the real spectrum complies with the current SUSY-type is expectrality in an unusual way. In this research paper, we conclude that the normalizable ground state exists for both H_L *and* H_R such that

$$E_{L0}^{-} = E_{R0}^{-} = -2$$

and the first excited state remains unmatched by any R-subscripted partner at

$E_L^+ = 0$ *as excepted*.The general supersymmetric partnership has been shown mediated by the "shape in variance" operators A^δ *and* B^δ . At any non integer $\propto = 0,$ the role of general creation and annihilation operators for a given, single Hamiltonian H^α has been shown played by their $\propto -$ dependent and $\beta -$ preserving products $A^+(\propto)$*and* $A(\propto)$ respectively. Intertwining operators plays most important role in SUSY quantum mechanics.

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