ON THE BASIS OF RELATIVE TYPE AND RELATIVE WEAK TYPE OF ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES

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Abstract: We introduced the idea of relative type and relative weak type of entire function of several complex variables with respect to another entire function of several complex variables and prove some result related to growth properties of it.

IndexTerms **- Entire functions, relative type, relative weak type, several complex variables.**

INTRODUCTION, DEFINITIONS AND NOTATIONS

Suppose f be an entire function of several complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2, \dots z_n) : |z_i| \le r_i, i = 1,2,\dots,n \quad \forall\ r_1 \ge 0, r_2 \ge 0, \dots, r_n \ge 0\}$$

and $M_f(r_1, r_2, \dots, r_n) = \max\{|f(z_1, z_2, \dots, z_n)| : |z_i| \le r_i, i = 1,2,\dots,n\}$.

Then in the light of maximum principal and Hartogs's theorem $\{[7], p.2, p.51\}$, $M_f(r_1, r_2, \dots, r_n)$ is an increasing function of $r_1, r_2, \dots, r_n$. For any two entire functions f and g of two complex variables, the ratio $\frac{M_f(r_1,r_2,\dots,r_n)}{M_g(r_1,r_2,\dots,r_n)}$ $as\ r_1, r_2, \dots, r_n \to \infty$ is called the growth of f with respect to g. Taking this into account, the following definition is well known:

DEFINITION 1. ([7], p.339, see also [1]) The order ${}_{vn}\rho_f$ and lower order ${}_{vn}\lambda_f$ of an entire function $f(z_1, z_2, \dots z_n)$ are defined as

$${}_{v_n}\rho_f = \limsup_{r_1,r_2,\dots,r_n\to\infty} \frac{\log\log M_f(r_1,r_2,\dots,r_n)}{\log M_{\exp(z_1 z_2 \dots z_n)}(r_1,r_2,\dots,r_n)} \quad \text{and} \quad {}_{v_n}\lambda_f = \liminf_{r_1,r_2,\dots,r_n\to\infty} \frac{\log\log M_f(r_1,r_2,\dots,r_n)}{\log M_{\exp(z_1 z_2 \dots z_n)}(r_1,r_2,\dots,r_n)}$$

We see that the order ${}_{v_n}\rho_f$ and lower order ${}_{v_n}\lambda_f$ of an entire function $f(z_1, z_2, \dots z_n)$ is defined in terms of the growth of $f(z_1, z_2, \dots z_n)$ with respect to the exponential function $exp(z_1, z_2, \dots z_n)$. The rate of growth of an entire function generally depends upon the order (lower order) of it. The entire function with higher order is of faster growth than that of lesser order. But if orders of two entire functions are the same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their types and thus one can define type of an entire function $f(z_1, z_2, \dots z_n)$ denoted by ${}_{v_n}\sigma_f$ in the following way.

DEFINITION 2. [6] *The* type ${}_{v_n}\sigma_f$ of an entire function $f(z_1, z_2, \dots z_n)$ is defined as

$${}_{v_n}\sigma_f = \limsup_{r_1,r_2,\dots,r_n\to\infty} \frac{\log M_f(r_1,r_2,\dots,r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\sigma_f}}, 0 < {}_{v_n}\sigma_f < \infty$$

Similarly, the lower type ${}_{v_n}\sigma_f$ of an entire function $f(z_1, z_2, \dots z_n)$ may be defined as

$${}_{v_n}\overline{\sigma}_f = \liminf_{r_1,r_2,\dots,r_n\to\infty} \frac{\log M_f(r_1,r_2,\dots,r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\rho_f}}, 0 < {}_{v_n}\sigma_f < \infty$$

Similarly in order to determine the relative growth of two entire functions of several complex variables having same non-zero finite lower order one may introduce the concept of weak type ${}_{v_n}\tau_f$ of $f(z_1, z_2, \dots z_n)$ of finite positive lower order ${}_{v_n}\lambda_f$ which is as follows

DEFINITION 3. [6] The weak type ${}_{v_n}\tau_f$ of an entire function $f(z_1, z_2, \dots, z_n)$ of finite positive *lower* order ${}_{v_n}\lambda_f$ is defined by

$${}_{v_n}\tau_f = \liminf_{r_1,r_2,\dots,r_n\to\infty} \frac{\log M_f(r_1,r_2,\dots,r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\lambda_f}}, 0 < {}_{v_n}\lambda_f < \infty$$

Likewise, one may define the growth indicator ${}_{v_n}\bar{\tau}_f$ of an entire function $f(z_1, z_2, \dots z_n)$ of finite positive lower order ${}_{v_n}\lambda_f$ in the following way

$${}_{v_n}\bar{\tau}_f = \limsup_{r_1, r_2, \dots, r_n \to \infty} \frac{\log M_f(r_1, r_2, \dots, r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\lambda_f}}, 0 < {}_{v_n}\lambda_f < \infty$$

Bernal (see [2], [3]) introduced the definition of relative order between two entire functions of single variable. During the past decades, several authors (see [8],[9],[10],[11]) made closed investigations on the properties of relative order of entire functions of single variable. Using the idea of Bernal's relative order (see [2], [3]) of entire functions of single variable, Banerjee and Datta [4] introduced the definition of relative order of entire functions of two complex variables to avoid comparing growth just with $exp(z_1, z_2, \dots z_n)$ which is as follows.

$${}_{v_n}\rho_g = \inf\{\mu > 0, M_f(r_1, r_2, \dots, r_n) < M_g(r_1^{\mu}, r_2^{\mu}, \dots, r_n^{\mu});\ r_i \geq R(\mu),\ \ i = 1,2, \dots, n\}$$

$$= \limsup_{r_1, r_2, \dots, r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{log(r_1 r_2 \dots r_n)}$$

where g is also an entire function holomorphic in the closed polydisc

$$\mathrm{U} = \{(z_1, z_2, \dots, z_n) : |z_i| \leq r_i, i = 1,2, \dots, n \,\forall\, r_i \geq 0\}$$

and the definition coincides with the classical one if $g(z_1, z_2, \dots, z_n) = exp(z_1 z_2 \dots z_n)$.

Likewise, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g$ as follows

$${}_{v_n}\lambda_g = \liminf_{r_1, r_2, \dots, r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{log(r_1 r_2 \dots r_n)}$$

Now in the case of relative order of entire functions of two complex variables, it therefore seems reasonable to define suitably the relative type and relative weak type respectively in order to compare the relative growth of two entire functions of two complex variables having same non zero finite relative order or relative lower order with respect to another entire function of two complex variables. Recently Datta introduced such definitions which are as follows.

DEFINITION 4. *Let* $f(z_1, z_2, \dots z_n)$ *and* $g(z_1, z_2, \dots z_n)$ be any two entire functions such that $0 < {}_{v_n}\sigma_g < \infty$. *Then the* relative type ${}_{v_n}\sigma_g(f)$ of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots z_n)$ is defined as.

$${}_{v_n}\sigma_g = \inf\left\{k > 0, M_f(r_1, r_2, \dots, r_n) < M_g\left(k r_1^{{}_{v_n}\rho_g}, k r_2^{{}_{v_n}\rho_g}, \dots, k r_n^{{}_{v_n}\rho_g}\right)\right\}$$

for all sufficiently large values of r_1, r_2, *and* r_n.

Equivalent formula for ${}_{v_n}\sigma_g$ is,

$${}_{v_n}\sigma_g = \limsup_{r_1, r_2, \dots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\rho_g}}$$

Likewise, one can define the relative lower type of an entire function $f(z_1, z_2, \dots z_n)$ to $g(z_1, z_2, \dots z_n)$ denoted by ${}_{v_n}\bar{\sigma}_g$ as follows

$${}_{v_n}\bar{\sigma}_g = \liminf_{r_1, r_2, \dots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\rho_g}}, 0 < {}_{v_n}\rho_g < \infty.$$

DEFINITION 5. [6] The relative weak type ${}_{v_n}\tau_g(f)$ of an entire function $f(z_1, z_2, \dots, z_n)$ with respect to another entire function $g(z_1, z_2, \dots, z_n)$ having finite positive relative lower order ${}_{v_n}\lambda_g(f)$ is defined as

$${}_{v_n}\tau_g(f) = \liminf_{r_1, r_2, \dots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\lambda_g}}$$

Also one may define the growth indicator ${}_{v_n}\bar{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way

$${}_{v_n}\bar{\tau}_g(f) = \limsup_{r_1, r_2, \dots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \dots, r_n)}{[(r_1 r_2 \dots r_n)]^{{}_{v_n}\lambda_g}}, 0 < {}_{v_n}\lambda_g < \infty$$

Considering $g(z_1, z_2, \dots, z_n) = exp(z_1 z_2 \dots z_n)$ one may easily verify that Definition 4 and Definition 5 coincide with Definition 2 and Definition 3 respectively.

In the paper we investigate some relative growth properties of entire functions of several complex variables with respect to another entire function of several complex variables on the basis of relative type and relative weak type of several complex variables. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [7].

LEMMA . [5] Let f and g be any two entire functions of several complex variables then

$$\frac{_{v_n}\lambda_f}{_{v_n}\rho_g} \leq _{v_n}\lambda_g(f) \leq \min\left\{\frac{_{v_n}\lambda_f}{_{v_n}\lambda_g}, \frac{_{v_n}\rho_f}{_{v_n}\rho_g}\right\} \leq \max\left\{\frac{_{v_n}\lambda_f}{_{v_n}\lambda_g}, \frac{_{v_n}\rho_f}{_{v_n}\rho_g}\right\} \leq _{v_n}\rho_g \leq \frac{_{v_n}\rho_f}{_{v_n}\lambda_g}$$

THEOREMS

In this section we present the main results of the paper.

THEOREM 1. Let $f(z_1, z_2, \dots, z_n)$ *and* $g(z_1, z_2, \dots, z_n)$ be any two entire functions with

$0 \leq _{v_n}\lambda_f \leq _{v_n}\rho_f < \infty$ and $0 \leq _{v_n}\lambda_g \leq _{v_n}\rho_g < \infty$. Then

$$\max\left\{\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\tau_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, \left[\frac{_{v_n}\sigma_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}\right\} \leq _{v_n}\sigma_g(f) \leq \min\left\{\left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\tau_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, \left[\frac{_{v_n}\sigma_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}\right\}$$

and

$$\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}} \leq _{v_n}\overline{\sigma}_g(f) \leq \min\left\{\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\sigma_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\tau_f}{_{v_n}\tau_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\tau_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}\right\}$$

PROOF: From the definitions of $_{v_n}\sigma_f$ and $_{v_n}\overline{\sigma}_f$ we have for all sufficiently large values of $r_1, r_2, \dots, and\ r_n$ that

$$M_f(r_1, r_2, \dots, r_n) \leq exp\left\{\left(_{v_n}\sigma_f + \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_f}\right\} \quad (1)$$

$$M_f(r_1, r_2, \dots, r_n) \geq exp\left\{\left(_{v_n}\overline{\sigma}_f - \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_f}\right\}$$

(2)

and also for a sequence of values of $r_1, r_2, \dots, and\ r_n$ tending to infinity, we get that

$$M_f(r_1, r_2, \dots, r_n) \geq exp\left\{\left(_{v_n}\sigma_f - \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_f}\right\} \quad (3)$$

$$M_f(r_1, r_2, \dots, r_n) \leq exp\left\{\left(_{v_n}\overline{\sigma}_f + \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_f}\right\} \quad (4)$$

Similarly from the definitions of $_{v_n}\sigma_g$ and $_{v_n}\overline{\sigma}_f$, it follows for all sufficiently large values of $r_1, r_2, \dots, r_n$ that

$$M_g(r_1, r_2, \dots, r_n) \leq exp\left\{\left(_{v_n}\sigma_g + \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_g}\right\}$$

i.e. , $(r_1, r_2, \dots, r_n) \leq M_g^{-1}\left[exp\left\{\left(_{v_n}\sigma_g + \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_g}\right\}\right]$

and $M_g^{-1}(r_1, r_2, \dots, r_n) \geq \left[\left(\frac{\log(r_1 r_2 \dots r_n)}{_{v_n}\rho_g + \varepsilon}\right)^{\frac{1}{_{v_n}\rho_g}}\right]$

(5)

Thus $M_g(r_1, r_2, \dots, r_n) \geq exp\left\{\left(_{v_n}\overline{\sigma}_g - \varepsilon\right)[(r_1 r_2 \dots r_n)]^{_{v_n}\rho_g}\right\}$

i.e., $M_g^{-1}(r_1, r_2, \dots, r_n) \leq \left[\left(\frac{\log(r_1 r_2 \dots r_n)}{_{v_n}\overline{\rho}_g - \varepsilon}\right)^{\frac{1}{_{v_n}\rho_g}}\right]$ (6)

and for a sequence of values of $r_1, r_2, \dots, r_n$ tending to infinity, we obtain that

$$M_g(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\sigma_g - \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\rho_g}\right\}$$

Thus $M_g^{-1}(r_1, r_2, \ldots, r_n) \leq \left[\left(\frac{\log(r_1 r_2 \ldots r_n)}{{}_{v_n}\rho_g - \varepsilon}\right)^{\frac{1}{{}_{v_n}\rho_g}}\right]$ (7)

$$M_g(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\overline{\sigma}_g + \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\rho_g}\right\}$$

i.e., $M_g^{-1}(r_1, r_2, \ldots, r_n) \geq \left[\left(\frac{\log(r_1 r_2 \ldots r_n)}{{}_{v_n}\overline{\rho}_g - \varepsilon}\right)^{\frac{1}{{}_{v_n}\rho_g}}\right]$ (8)

From the definitions of ${}_{v_n}\tau_f$ and ${}_{v_n}\overline{\tau}_f$, we have for all sufficiently large values of $r_1, r_2, \ldots, r_n$ that

$$M_f(r_1, r_2, \ldots, r_n) \leq exp\left\{\left({}_{v_n}\overline{\tau}_f + \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_f}\right\}$$

(9)

$$M_f(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\tau_f - \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_f}\right\}$$ (10)

and also for a sequence of values of $r_1, r_2, \ldots, and\ r_n$ tending to infinity, we get that

$$M_f(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\overline{\tau}_f - \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_f}\right\}$$

(11)

$$M_f(r_1, r_2, \ldots, r_n) \leq exp\left\{\left({}_{v_n}\tau_f + \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_f}\right\}$$ (12)

Similarly from the definitions of ${}_{vn}\tau_g$ and ${}_{vn}\bar{\tau}_g$, it follows for all sufficiently large values of $r_1, r_2, \ldots, r_n$ that

$$M_g(r_1, r_2, \ldots, r_n) \leq exp\left\{\left({}_{v_n}\overline{\tau}_g + \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_g}\right\}$$

i.e., $(r_1, r_2, \ldots, r_n) \leq M_g^{-1}\left[exp\left\{\left({}_{v_n}\overline{\tau}_g + \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_g}\right\}\right]$

and $M_g^{-1}(r_1, r_2, \ldots, r_n) \geq \left[\left(\frac{\log(r_1 r_2 \ldots r_n)}{{}_{v_n}\overline{\tau}_g + \varepsilon}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right]$

(13)

Thus $M_g(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\tau_g - \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_g}\right\}$

i.e., $M_g^{-1}(r_1, r_2, \ldots, r_n) \leq \left[\left(\frac{\log(r_1 r_2 \ldots r_n)}{{}_{v_n}\tau_g - \varepsilon}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right]$

(14)

and for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity, we obtain that

$$M_g(r_1, r_2, \ldots, r_n) \geq exp\left\{\left({}_{v_n}\overline{\tau}_g - \varepsilon\right) [(r_1 r_2 \ldots r_n)]^{{}_{v_n}\lambda_g}\right\}$$

Thus $$M_g^{-1}(r_1,r_2,\dots,r_n) \le \left[\left(\frac{\log(r_1r_2\dots r_n)}{{}_{v_n}\bar{\tau}_g - \varepsilon}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right]$$

(15)

$$M_g(r_1,r_2,\dots,r_n) \le exp\left\{\left({}_{v_n}\tau_g + \varepsilon\right)[(r_1r_2\dots r_n)]^{{}_{v_n}\lambda_g}\right\}$$

i.e., $$M_g^{-1}(r_1,r_2,\dots,r_n) \ge \left[\left(\frac{\log(r_1r_2\dots r_n)}{{}_{v_n}\tau_g + \varepsilon}\right)^{\frac{1}{{}_{v_n}\lambda_g}}\right]$$

(16)

Now from (3) and in view of (13), we get for a sequence of values of $r_1, r_2,\dots, r_n$ tending to infinity that

$$\frac{M_g^{-1}M_f(r_1,r_2,\dots,r_n)}{[r_1r_2\dots r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g}}} \ge \left[\frac{\left({}_{v_n}\sigma_f - \varepsilon\right)}{\left({}_{v_n}\bar{\tau}_f + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

Since in view of lemma

$${}_{v_n}\sigma_g(f) \ge \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\bar{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(17)

Similarly from (2) and in view of (16) it follows for a sequence of values of $r_1, r_2,\dots, r_n$ tending to infinity that

$$\frac{M_g^{-1}M_f(r_1,r_2,\dots,r_n)}{[r_1r_2\dots r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g}}} \ge \left[\frac{\left({}_{v_n}\bar{\sigma}_f - \varepsilon\right)}{\left({}_{v_n}\tau_g + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

Since in view of lemma

$${}_{v_n}\sigma_g(f) \ge \left[\frac{{}_{v_n}\bar{\sigma}_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}} \qquad (18)$$

Again in view of (14), we have from (9) for all sufficiently large values of $r_1, r_2,\dots, r_n$ and in view of lemma

$$\frac{M_g^{-1}M_f(r_1,r_2,\dots,r_n)}{[r_1r_2\dots r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}}} \ge \left[\frac{\left({}_{v_n}\bar{\tau}_f + \varepsilon\right)}{\left({}_{v_n}\tau_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

$${}_{v_n}\sigma_g(f) \ge \left[\frac{{}_{v_n}\bar{\tau}_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(19)

Again in view of (6), we have from (1) for all sufficiently large values of $r_1, r_2,\dots, r_n$ that

$$M_g^{-1}M_f(r_1,r_2,\dots,r_n) \le M_g^{-1}\left[exp\left\{\left({}_{v_n}\sigma_f + \varepsilon\right)[(r_1r_2\dots r_n)]^{{}_{v_n}\rho_f}\right\}\right]$$

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{v_n\lambda_f}{v_n\rho_g}}} \leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\bar{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}} \tag{20}$$

Since in view of lemma

$$_{v_n}\sigma_g(f) \leq \left[\frac{_{v_n}\sigma_f}{_{v_n}\bar{\sigma}_g}\right]^{\frac{1}{v_n\rho_g}} \tag{21}$$

Further in view of (6), we have from (9) for all sufficiently large values of $r_1, r_2,..., r_n$ that

$$M_g{}^{-1}M_f(r_1,r_2,\ldots,r_n) \leq M_g{}^{-1}\left[exp\left\{\left({}_{v_n}\bar{\tau}_f+\varepsilon\right)\,[(r_1r_2\ldots r_n)]^{v_n\lambda_f}\right\}\right]$$

That is,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{v_n\lambda_f}{v_n\rho_g}}} \leq \left[\frac{\left({}_{v_n}\bar{\tau}_f+\varepsilon\right)}{\left({}_{v_n}\bar{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}} \tag{22}$$

Since in view of lemma, we get that $\frac{_{v_n}\lambda_f}{_{v_n}\rho_g} \leq {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$_{v_n}\rho_g^{L^*}(f) \leq \left[\frac{_{v_n}\bar{\tau}_f}{_{v_n}\bar{\rho}_g}\right]^{\frac{1}{v_n\rho_g}}$$

(23)

Thus the theorem follows from (17), (18), (19), (21) and (23).

Further from (2) and in view of (13), we get for all sufficiently large values of $r_1, r_2,..., r_n$ that

$$M_g^{-1}M_f(r_1,r_2,...,r_n) \leq M_g^{-1}\left[\exp\left\{\left({}_{v_n}\bar{\rho}_f-\varepsilon\right)[r_1r_2...r_n]^{v_n\rho_f}\right\}\right]$$

That is,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{v_n\rho_f}{v_n\lambda_g}}} \leq \left[\frac{\left({}_{v_n}\bar{\sigma}_f-\varepsilon\right)}{\left({}_{v_n}\bar{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{v_n\lambda_g}}$$

Since in view of lemma, we get that $\frac{_{v_n}\rho_f}{_{v_n}\lambda_g} \leq {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore we get that

$$_{v_n}\bar{\rho}_g(f) \leq \left[\frac{_{v_n}\bar{\sigma}_f}{_{v_n}\bar{\tau}_g}\right]^{\frac{1}{v_n\lambda_g}} \tag{24}$$

Also in view of (7), we get from (1) for a sequence of values of $r_1, r_2,..., r_n$ tending to infinity that

$$M_g^{-1}M_f(r_1,r_2,...,r_n) \leq M_g^{-1}\left[\exp\left\{\left({}_{v_n}\sigma_f+\varepsilon\right)[r_1r_2...r_n]^{v_n\rho_f}\right\}\right]$$

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{v_n\rho_f}{v_n\rho_g}}} \leq \left[\frac{\left({}_{v_n}\sigma_f+\varepsilon\right)}{\left({}_{v_n}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{v_n\rho_g}} \tag{25}$$

That is,

$$_{v_n}\bar{\rho}_g(f) \leq \left[\frac{_{v_n}\sigma_f}{_{v_n}\sigma_g}\right]^{\frac{1}{v_n\rho_g}}$$

(26)

Likewise from (4) and in view of (6), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, ..., r_n) \le M_g^{-1}\left[\exp\left\{\left({}_{v_n}\overline{\sigma}_f + \varepsilon\right)[r_1 r_2 ... r_n]^{{}_{v_n}\rho_f}\right\}\right]$$

(27)

We get,

$$\frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}}} \le \left[\frac{\left({}_{v_n}\overline{\sigma}_f + \varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

(28)

We get from (28) that,

$${}_{v_n}\overline{\rho}_g(f) \le \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_n}\rho_g}} \qquad (29)$$

Since in view of lemma, $\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g} \le {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore further in view of (15), we get from (9) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, \dots, r_n) \le M_g^{-1}\left[exp\left[\left({}_{v_n}\overline{\tau}_f + \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Then,

$$\frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}}} \le \left[\frac{\left({}_{v_n}\overline{\tau}_f + \varepsilon\right)}{\left({}_{v_n}\overline{\tau}_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(30)

As in view of lemma $\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g} \le {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$${}_{v_n}\overline{\rho}_g^{L^*}(f) \le \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}} \qquad (31)$$

Similarly from (12) and in view of (14), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, \dots, r_n) \le M_g^{-1}\left[exp\left[\left({}_{v_n}\tau_f + \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{{}_{v_n}\lambda_g}\right]\right]$$

Then,

$$\frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}}} \le \left[\frac{\left({}_{v_n}\tau_f + \varepsilon\right)}{\left({}_{v_n}\tau_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(32)

As in view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g} \le {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$${}_{v_n}\overline{\rho}_g(f) \le \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}} \qquad (33)$$

Again in view of (7), we get from (9) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f(r_1, r_2, \dots, r_n) \le M_g^{-1}\left[exp\left[\left({}_{v_n}\overline{\tau}_f + \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Then, $$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{_{v_n}\lambda_f^{L^*}}{_{v_n}\rho_g}}} \le \left[\frac{\left(_{v_n}\overline{\tau}_f+\varepsilon\right)}{\left(_{v_n}\sigma_g-\varepsilon\right)}\right]^{\frac{1}{_{v_n}\rho_g}}$$

(34)

As in view of lemma, $\frac{_{v_n}\lambda_f}{_{v_n}\sigma_g} \le {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$_{v_n}\overline{\rho}_g(f) \le \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}} \tag{35}$$

Similarly from (12) and in view of (6), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1}M_f(r_1,r_2,...,r_n) \le M_g^{-1}\left[exp\left[\left(_{v_n}\tau_f+\varepsilon\right)\left[r_1r_2...r_n\right]^{_{v_n}\lambda_f}\right]\right]$$

Then, $$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{_{v_n}\lambda_f}{_{v_n}\rho_g}}} \le \left[\frac{\left(_{v_n}\tau_f+\varepsilon\right)}{\left(_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{_{v_n}\rho_g}}$$

(36)

As in view of lemma, $\frac{_{v_n}\lambda_f}{_{v_n}\sigma_g} \le {}_{v_n}\rho_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$_{v_n}\overline{\rho}_g(f) \le \left[\frac{_{v_n}\tau_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}$$

(37)

Hence the second part of the theorem follows from (24), (26), (29), (31), (33), (35) and (37).

THEOREM 2. Let $f(z_1, z_2, ..., z_n)$ *and* $g(z_1, z_2, ..., z_n)$ be any two entire functions with $0 \le {}_{v_n}\lambda_f \le {}_{v_n}\rho_f < \infty$ and $0 \le {}_{v_n}\lambda_g \le {}_{v_n}\rho_g < \infty$

Then

$$\max\left\{\begin{matrix}\left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, & \left[\frac{_{v_n}\tau_f}{_{v_n}\tau_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, & \left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}} \\ \left[\frac{_{v_n}\sigma_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}}, & \left[\frac{_{v_n}\sigma_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, & \left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\tau_g}\right]^{\frac{1}{_{v_n}\lambda_g}}\end{matrix}\right\} \le {}_{v_n}\overline{\tau}_g(f) \le \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}$$

and

$$\max\left\{\left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\tau_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}, \left[\frac{_{v_n}\overline{\sigma}_f}{_{v_n}\overline{\tau}_g}\right]^{\frac{1}{_{v_n}\lambda_g}}\right\} \le {}_{v_n}\tau_g(f) \le \min\left\{\left[\frac{_{v_n}\tau_f}{_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{_{v_n}\rho_g}}, \left[\frac{_{v_n}\overline{\tau}_f}{_{v_n}\sigma_g}\right]^{\frac{1}{_{v_n}\rho_g}}\right\}$$

Proof: We obtain from (11) and (13), for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1}M_f\left(r_1,r_2,...,r_n\right) \ge M_g^{-1}\left[exp\left[\left(_{v_n}\overline{\tau}_f-\varepsilon\right)\left[r_1r_2...r_n\right]^{_{v_n}\lambda_f}\right]\right]$$

Then,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{_{v_n}\lambda_f}{_{v_n}\lambda_g}}} \ge \left[\frac{\left(_{v_n}\overline{\tau}_f-\varepsilon\right)}{\left(_{v_n}\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{_{v_n}\lambda_g}}$$

As in view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\sigma_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{{}_{v_n}\lambda_g(f)}} \geq \left[\frac{{}_{v_n}\bar{\tau}_f}{{}_{v_n}\bar{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

and

$${}_{v_n}\bar{\tau}_f \geq \left[\frac{{}_{v_n}\bar{\tau}_f}{{}_{v_n}\bar{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(38)

Further we obtain from (10) and (16), for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\tau_f - \varepsilon\right)\left[r_1 r_2 \ldots r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}}} \geq \left[\frac{\left({}_{v_n}\bar{\tau}_f - \varepsilon\right)}{\left({}_{v_n}\bar{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

As in view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\sigma_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{{}_{v_n}\lambda_g(f)}} \geq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

and

$${}_{v_n}\bar{\tau}_f \geq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(39)

Now from (3) and in view of (5), we get for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\sigma_f - \varepsilon\right)\left[r_1 r_2 \ldots r_n\right]^{{}_{v_n}\rho_f}\right]\right]$$

Therefore, $$\frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}}} \geq \left[\frac{\left({}_{v_n}\sigma_f - \varepsilon\right)}{\left({}_{v_n}\sigma_f + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}} \tag{40}$$

Also from (2) and in view of (8), it follows for a sequence of values of $r_1, r_2, \ldots, r_n$ ending to infinity that

$$M_g^{-1} M_f\left(r_1, r_2, \ldots, r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\bar{\sigma}_f - \varepsilon\right)\left[r_1 r_2 \ldots r_n\right]^{{}_{v_n}\rho_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}}} \geq \left[\frac{\left({}_{v_n}\bar{\sigma}_f - \varepsilon\right)}{\left({}_{v_n}\bar{\sigma}_g + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

(41)

As in view of lemma, $\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, from (40) we get that

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{v_n \lambda_g(f)}} \geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n \rho_g}}$$

Then,

$$_{v_n}\bar{\tau}_g \geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{v_n \rho_g}} \quad (42)$$

Similarly, we get from equation (41) that

$$_{v_n}\bar{\tau}_g \geq \left[\frac{{}_{v_n}\bar{\sigma}_f}{{}_{v_n}\bar{\sigma}_g}\right]^{\frac{1}{v_n \rho_g}}$$

(43)

In view of lemma, $\frac{v_n \rho_f}{v_n \rho_g} \leq_{v_n} \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

Likewise from (3) and in view of (13), we get for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f\left(r_1, r_2, ..., r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\sigma_f - \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{v_n \rho_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \rho_g}}} \geq \left[\frac{\left({}_{v_n}\sigma_f - \varepsilon\right)}{\left({}_{v_n}\bar{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{v_n \lambda_g}} \quad (44)$$

In view of lemma, $\frac{v_n \rho_f}{v_n \lambda_g} \leq_{v_n} \lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{v_n \lambda_g(f)}} \geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\bar{\tau}_g}\right]^{\frac{1}{v_n \lambda_g}}$$

Then,

$$_{v_n}\bar{\tau}_g \geq \left[\frac{{}_{v_n}\sigma_f}{{}_{v_n}\bar{\tau}_g}\right]^{\frac{1}{v_n \lambda_g}}$$

(45)

From (2) and in view of (16), we get for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1} M_f\left(r_1, r_2, ..., r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\bar{\sigma}_f - \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{v_n \rho_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1} M_f(r_1, r_2, ..., r_n)}{[r_1 r_2 ... r_n]^{\frac{v_n \rho_f}{v_n \lambda_g}}} \geq \left[\frac{\left({}_{v_n}\bar{\sigma}_f - \varepsilon\right)}{\left({}_{v_n}\tau_g - \varepsilon\right)}\right]^{\frac{1}{v_n \lambda_g}} \quad (46)$$

Then,

$$_{v_n}\bar{\tau}_g \geq \left[\frac{{}_{v_n}\bar{\sigma}_f}{{}_{v_n}\tau_g}\right]^{\frac{1}{v_n \lambda_g}} \quad (47)$$

Again from (6) and (9), we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g^{-1} M_f\left(r_1, r_2, ..., r_n\right) \leq M_g^{-1}\left[exp\left[\left({}_{v_n}\bar{\tau}_f + \varepsilon\right)\left[r_1 r_2 ... r_n\right]^{v_n \lambda_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g}}} \leq \left[\frac{\left({}_{v_n}\overline{\tau}_f+\varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

In view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{{}_{v_n}\lambda_g(f)}} \leq \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

That is,

$${}_{v_n}\overline{\tau}_g \leq \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_n}\rho_g}} \tag{48}$$

Thus the theorem follows from (38), (39), (42), (43), (45), (47) and (48).

Further from (10) and in view of (13), we get for all sufficiently large values of $r_1,r_2,...,r_n$ that

$$M_g{}^{-1}M_f\left(r_1,r_2,...,r_n\right) \geq M_g{}^{-1}\left[exp\left[\left({}_{v_n}\tau_f-\varepsilon\right)\left[r_1r_2...r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}}} \leq \left[\frac{\left({}_{v_n}\tau_f-\varepsilon\right)}{\left({}_{v_n}\overline{\tau}_g+\varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

In view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{{}_{v_n}\lambda_g(f)}} \leq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

That is,

$${}_{v_n}\tau_g \geq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(49)

Again from (2) and in view of (5), we get for all sufficiently large values of $r_1,r_2,...,r_n$ that

$$M_g{}^{-1}M_f\left(r_1,r_2,...,r_n\right) \geq M_g{}^{-1}\left[exp\left[\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)\left[r_1r_2...r_n\right]^{{}_{v_n}\rho_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}}} \geq \left[\frac{\left({}_{v_n}\overline{\sigma}_f-\varepsilon\right)}{\left({}_{v_n}\sigma_g+\varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

(50)

In view of lemma, $\frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary, we get from (50) that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{M_g^{-1}M_f(r_1,r_2,...,r_n)}{[r_1r_2...r_n]^{{}_{v_n}\lambda_g(f)}} \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

That is,

$${}_{v_n}\tau_g \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

(51)

Again from (2) and in view of (13), we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g^{-1}M_f\left(r_1, r_2, ..., r_n\right) \geq M_g^{-1}\left[exp\left[\left({}_{v_n}\overline{\sigma}_f - \varepsilon\right)\left[r_1r_2...r_n\right]^{{}_{v_n}\rho_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{\left[r_1r_2...r_n\right]^{\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g}}} \geq \left[\frac{\left({}_{v_n}\overline{\sigma}_f - \varepsilon\right)}{\left({}_{v_n}\overline{\tau}_g + \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

In view of lemma, $\frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{\left[r_1r_2...r_n\right]^{{}_{v_n}\lambda_g(f)}} \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

That is,

$${}_{v_n}\tau_g \geq \left[\frac{{}_{v_n}\overline{\sigma}_f}{{}_{v_n}\overline{\tau}_g}\right]^{\frac{1}{{}_{v_n}\lambda_g}}$$

(52)

We get from (7) and (9) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1}M_f\left(r_1, r_2, ..., r_n\right) \leq M_g^{-1}\left[exp\left[\left({}_{v_n}\overline{\tau}_f + \varepsilon\right)\left[r_1r_2...r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{\left[r_1r_2...r_n\right]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g}}} \geq \left[\frac{\left({}_{v_n}\overline{\tau}_f + \varepsilon\right)}{\left({}_{v_n}\sigma_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

As in view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{\left[r_1r_2...r_n\right]^{{}_{v_n}\lambda_g(f)}} \leq \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

That is,

$${}_{v_n}\tau_g \leq \left[\frac{{}_{v_n}\overline{\tau}_f}{{}_{v_n}\sigma_g}\right]^{\frac{1}{{}_{v_n}\rho_g}} \quad (53)$$

Similarly, from (12) and in view of (6), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$M_g^{-1}M_f\left(r_1, r_2, ..., r_n\right) \leq M_g^{-1}\left[exp\left[\left({}_{v_n}\tau_f + \varepsilon\right)\left[r_1r_2...r_n\right]^{{}_{v_n}\lambda_f}\right]\right]$$

Therefore,

$$\frac{M_g^{-1}M_f(r_1, r_2, ..., r_n)}{\left[r_1r_2...r_n\right]^{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g}}} \leq \left[\frac{\left({}_{v_n}\tau_f + \varepsilon\right)}{\left({}_{v_n}\overline{\sigma}_g - \varepsilon\right)}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

In view of lemma, $\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f)$ and as $\varepsilon(>0)$ is arbitrary

$$\liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{M_g^{-1} M_f(r_1, r_2, \ldots, r_n)}{[r_1 r_2 \ldots r_n]^{{}_{v_n}\lambda_g(f)}} \leq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

That is,

$${}_{v_n}\tau_g \leq \left[\frac{{}_{v_n}\tau_f}{{}_{v_n}\overline{\sigma}_g}\right]^{\frac{1}{{}_{v_n}\rho_g}}$$

(54)

Hence the second part of the theorem follows from (49), (51), (52), (53) and (54).

COROLLARY. Let f *and* g be any two entire functions of several complex variables such that g is of regular growth: Then

$${}_{v_n}\lambda_g(f) = \frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \text{ and } {}_{v_n}\rho_g(f) = \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}$$

In addition, if ${}_{v_n}\rho_f = {}_{v_n}\rho_g$; then ${}_{v_n}\rho_g(f) = {}_{v_n}\lambda_f = 1$

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