A FOUCAULT PENDULUM REPRESENTATION OF CHARGED PARTICLE'S MOTION IN ROTATING QUADRUPOLE POTENTIAL

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Abstract: Quadrupole Potential in a plane forms a kind of saddle potential for the charged particle in that plane. The potential in that plane is a hyperbolic function; increases in one dimension and decreases in the other. As a result, the electric field is inward in one dimension and outward in the other. The particle's motion in such a field configuration cannot have a stable equilibrium. It is seen that the quadrupole potential can be rotated in transverse plane and the particle's motion can be stabilized over time, for sufficiently larger frequencies of rotation [1, 2]. The transverse motion of a charged particle in such rotating quadrupole potential exhibits micro-circular motion and a prograde precision in larger scale. The motion of the guiding centre of the trajectory can be represented with an equation similar to that of a Foucault's pendulum.

Index Terms **– Rotating Quadrupole Potential, Brouwer's Equation, Foucault Pendulum, Charged Particle Dynamics.**

I. INTRODUCTION

In a stabilized motion of a particle in a rapidly rotating planar saddle potential, a precessional motion is also observed, which has been shown due to a Coriolis-like force caused by the rotation of the potential. This is a unique example where such a force arises in an inertial reference frame. The equations of motion in such system are the similar to the equations derived in 1918 by Brouwer (1881–1966), while considering the stability of a heavy particle on a rotating slippery surface. His famous equations, $\ddot{x} + x\cos 2\omega t + y\sin 2\omega t = 0$, $\ddot{y} + x\sin 2\omega t - y\cos 2\omega t = 0$, also govern the motion of a unit mass in a plane under the influence of a potential force given by the rotating saddle potential—the potential whose graph is obtained by rotating the graph $z = \frac{1}{2}(x^2 - y^2)$ around the z-axis with angular velocity ω. A good example is the motion of a charged particle in a rotating electrostatic quadrupole potential. The motion of a particle on such a saddle is stabilized for all sufficiently high ω. The particle trapped in the rotating saddle exhibits a prograde precession in the laboratory frame, i.e., the particle moves along an elongated trajectory that in itself slowly rotates in the laboratory frame with the angular velocity ω_p in the same sense as the saddle. It has been demonstrated that the rapid rotation of the saddle potential creates a weak Lorentz-like (or Coriolis-like) force, in addition to an effective stabilizing potential, all in the inertial frame [1]. It is also shown in [3] that a simplified equation can be found for the guiding center of the trajectory that coincides with the equation of the Foucault's pendulum. In this sense, a particle trapped in the symmetric rotating saddle trap is, effectively, a Foucault's pendulum, but in the inertial frame. In this paper, a similar approach is followed in case of charged particle motion in rotating quadrupole potential. The method will be of use while designing such devices in charged particle beam transfer lines used in particle accelerator systems or in ion traps. In charged particle beam transport systems, electrostatic of magnetic quadrupoles are used to keep the beam confined in transverse direction or to focus the beam. Inherently, a quadrupole field configuration cause focusing in one transverse plane and defocusing in the other transverse plane. A sequence of focusing and defocusing quadrupole field causes effective focusing of beam in both the transverse planes. Strong focusing, i.e., alternating focusing and defocusing fields, increases largely by exploring field rotations about the optic axis. Literatures are found on electric or magnetic quadrupoles rotating helically in space around the optic axis [3, 10]. Field rotating in time, rather than space, has also been discussed and it has been suggested that such strong focusing method can effectively be explored in beam line applications, particularly in radiofrequency quadrupole accelerator system.

II. **Motion of charged particle in Rotating Quadrupole potential**

We use a Cartesian coordinate system with a preferential longitudinal axis (z) along the beam transport line. Also we consider the transverse dimension of the device is much less than the longitudinal dimension, as it is generally in case of a beam transport device like electrostatic quadrupoles. The static scalar potential, follows the two dimensional Laplace equation in transverse (x, y) plane $\nabla_{x,y}{}^2\psi = 0$. The quadrupole term of the general solution is given by,

$$\psi(x, y) = a(x^2 - y^2) + 2bxy \qquad (1)$$

Now we consider the case where the scalar potential function is to be rotated around the z-axis. It can be done with a time varying *normal* $a(x^2 - y^2)$ and *skew* $2bxy$ configuration superimposed on each other, with a phase difference of $\pi/2$. The time varying rotatin potential function can be written as,

$$\psi(x, y, t) = a\cos(\omega t)\ (x^2 - y^2) + 2b\sin(\omega t)\ xy \qquad (2)$$

where angular frequency $\omega = 2\pi/T$, T being the time period of the sinusoidal potential.

Hence the equations of motion are as follows:

$$\ddot{x} = -\frac{q}{\gamma m_0}(2ax\ \cos(\omega t) + 2by\ \sin(\omega t)) \qquad (3)$$

$$\ddot{y} = \frac{q}{\gamma m_0}(-2ax\ \sin(\omega t) + 2by\ \cos(\omega t)) \qquad (4)$$

We may normalize the constants as $\frac{2aq}{\gamma m_0} = \frac{2bq}{\gamma m_0} = 1$,

$$\ddot{x} + x\ \cos(\omega t) + y\ \sin(\omega t) = 0 \qquad (5)$$

$$\ddot{y} + x\ \sin(\omega t) - y\ \cos(\omega t) = 0 \qquad (6)$$

The solutions of these equations in parametric form are shown in figure 1. The stable motion for all higher values is shown in figure 2. The particle moves on a stretched curved trajectory that itself rotates in the same sense as the rotating field.

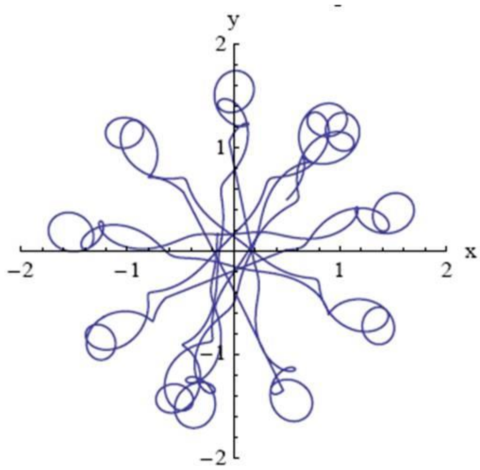

Figure 1: Trajectories of charged particle under rotating quadrupole potential. For $\omega = 3$ motion is stable or confined in the transverse direction. The particle executes a small scale circular micro-motion and a large scale circular secular motion.

III. Transformation

Now we may look in to the potential from a coordinate system (x', y') rotated by an angle φ say. Applying the transformation $x = x' \cos\varphi - y' \sin\varphi$ and $y = x' \sin\varphi + y' \cos\varphi$, we get,

$$\psi(x', y') = \left(x'^2 - y'^2\right)(a\cos 2\varphi + b\sin 2\varphi) + 2x'y'(b\cos 2\varphi - a\sin 2\varphi)$$

Without losing the generality of the problem, we can choose a rotated coordinate system such that the skew term vanishes, i.e., $b\cos 2\varphi - a\sin 2\varphi = 0$. Considering a symmetric normal and skew component, i.e., $b = a$, we find $\tan 2\varphi = 1$. Or, $\varphi = \pi/8$. In this coordinate system, the potential can be represented by

$$\psi(x', y') = \frac{\left(x'^2 - y'^2\right)}{2} \qquad (7)$$

Thus, the charged particle in a rotating quadrupole potential, behaves like a particle on a rotating saddle potential. Now, the governing equations (5) & (6) can be written in vector form as

$$\ddot{\boldsymbol{X}} + S(\omega t)\,\boldsymbol{X} = 0 \qquad (8)$$

Where, $\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $S(\omega t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ \sin\omega t & -\cos\omega t \end{bmatrix}$

Kirillov and Levi has shown that the motion $X(t)$, governed by equation (8), can be assigned with a “guiding centre” or “hodograph”, given by $\boldsymbol{u} = \boldsymbol{X} - \frac{\varepsilon^2}{2} S(\omega t)(\boldsymbol{X} - \varepsilon J\dot{\boldsymbol{X}})$, where $\varepsilon = 1/\omega$ and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is counter-clock wise rotation by $\pi/2$. They have shown that for large ω the dynamical equation for guiding centre, neglecting the terms of $O(\varepsilon^4)$, is given by

$$\ddot{\boldsymbol{u}} - \frac{\varepsilon^3}{4} J\dot{\boldsymbol{u}} + \frac{\varepsilon^2}{4}\boldsymbol{u} = 0 \qquad (9)$$

This is in fact the equation of Foucault pendulum.

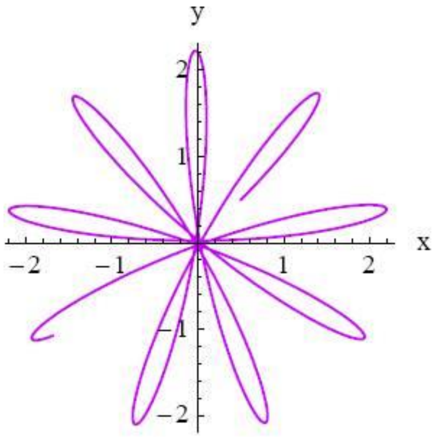

Figure 2: Motion of the 'guiding centre' governed by equation (9) for $\omega = 3$.

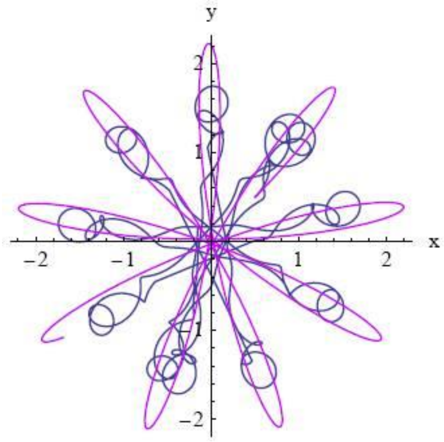

Figure 3: Motion of the 'guiding centre' superimposed on the motion of charged particle.

Figure 2 shows the motion of the 'guiding centre' and figure 3 shows the same superimposed on the motion of the charged particle. It shows that the large scale precession of the trajectory is depicted by the motion of an equivalent Foucault pendulum. This representation simplifies the calculations of gross dynamics of the charged particle in rotating quadrupole potential.

IV. CONCLUSION

Transverse dynamics of a charged particle in the rotating quadurpole potential is investigated. The equations of motion are similar to the famous Brouwer's equations for a particle on a rotating slippery surface. Parametric plots of numerical solution of the equations show that the motion is confined in transverse direction for all higher rotational frequencies. In case of stable motion, the trajectory in transverse plane is a combination of circular micromotion and a prograde precession. The motion can be represented by an equivalent Foucault Pendulum.

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