INNER PRODUCTS ON BANACH SPACES: SYMMETRIZABLE OPERATORS AND SOME GENERALIZATIONS

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Abstract : In order to generalize some aspects of the theory of symmetrizable operators on Hilbert spaces to the case of elements of the Banach algebra of all bounded linear operators on a Banach space, we consider here some results of the theory of inner products on Banach spaces.

Keywords : Banach spaces, symmetrizable operators, Hilbert spaces.

Introduction

In order to generalize some aspects of the theory of symmetrizable operators on Hilbert spaces to the case of elements of the Banach algebra of all bounded linear operators on a Banach space, we consider here some results of the theory of inner products on Banach spaces.

Let X be a complex Banach space and F be a continuous bilinear functional on X having the following properties:

1. $F(x, y) = F(y, x)^*$
2. $F(x, x) > 0$ for $x \neq 0$

Let $X \rightarrow F(x, x)^{1/2}$

and exactly as in the case of Hilbert spaces, we can show that this is a norm on X, which we denote by $|*|$.

The continuity of Fimplies that there is a constant $k > 0$ such that for all $x \in X$,

$|x| \leq k\|x\|$

where $\|*\|$ denotes the original norm of X.

Since the space $(x, |*|)$ has many of the properties of Hilbert spaces, except possibly the completeness (of course, with respect to $|*|$), we can define with respect to ten bilinear form F the notions hermitian operator, normal and unitary operator, etc.

Thus we are led naturally to the problem of the existence of inner products on Banach spaces. We mention that the inner products on Banach spaces were used by Lax (1954) in his study of symmetrizable operators on Banach spaces. Also Olagunju (Murphy and West (1972) has given an example of a Banach space which does not possess an inner product $<, >$ satisfying the following axioms:

1. $<x, x> > 0$ and $<x, x> = 0$ if and only if $x = 0$
2. $<x, y> = <y, x>^*$
3. $<ax + by, z> = a<x, z> + b<y, z>$
4. $<x, x>$ s $k^2\|x\|^2$

for all $x, y, z \in X$ and $a, b \in C$ and for some fixed constant k.

Example 1. Let K be a compact in R^m and C(K) be the set of all complex-valued continuous functions on K. With the sup norm this is a Banach space and the sup norm is denoted by $\|*\|$.

We can define, as usually, the inner product by

$$< f, g >= \int_K f(x)g(x)^* dx$$

where dx is the Lebesgue measure on R^m.

It is not difficult to see that this inner product satisfies all the above axioms.

Example 2. Let B, be the space of all bounded complex valued functions on I = [0, 1], with the norm

$$f \rightarrow \|f\| = \sup_t |f(t)|$$

For each $t_0 \in I$ we consider the function

$$x_{t_0}(t) = \begin{cases} 1 & t = t_0 \\ 0 & t \neq t_0 \end{cases}$$

which is obvious in B_1.

Suppose now that B_1 has an inner product satisfying the above axioms.

We show that in this case we obtain a contradiction.

For a > 0, we define

$Ta = \{t \in I | <x_1, x_1> > a\}$

First we show that for at least one n, $T_{1/n}$ is an infinite set.

Indeed since $<x, x> = 0$ implies $x = 0$, we have that $I = U_n T_{1/n}$ and if for all n, $T_{1/n}$ is a finite set, we obtain that I is countable and this is a contradiction.

Let n_0 be the integer for which $T = T_{1/n}$ is an infinite set and if

$t_1, t_2 \in I, t_1 \neq t_2$,

then we set

$$y_1 = x_{t_1} \qquad y_2 = x_{t_2} e^{is_2}$$

where s_2 is an appropriate real number such that

$$< x_{t_1}, x_{t_2} > e^{is_2} \geq 0.$$

We define

$z_1 = y_1$, $z_2 = y1 + 2^{-1/2}y_2$

Suppose now that Z_1, ..., Z_{n-1} are given.

Let $t_n \in T$, $t_n \neq t_i$ for i = 1, 2, ..., n - 1.

We take s_n such that

$$y_n = e^{is}\, nx_{t_n} \qquad < z_{n-2}, y_n > \geq 0.$$

and we define

$z_n = z_{n-1} + n^{-1/2}y_n$

Since t_1 are distinct points,

we have that

$$\|z_{n+1} - z_m\| = \left\|(n+1)^{-\frac{1}{2}} e^{is}(n+1)x_{1_{n+1}} + n^{-\frac{1}{2}} x_{t_n} e^{is} n + \ldots + m^{-\frac{1}{2}} e^{is} m x_{t_m}\right\| = (n+1)^{-\frac{1}{2}}$$

for m < n.

From this it is clear that (z_n) is a Cauchy sequence and thus we find an element z e B, such that

lim z_n = z

Axiom 4 of the inner product implies that

lim <z_n, z_n> = <z, z>

Since we have the relation

<z_n, z_n> <y_1, y_1> + ... + <y_n, y_n>

we obtain a contradiction.

The above inequality can be proved by an induction argument. The assertion is true for n = 1.

Suppose that it is true for all i sn - 1 and we show that it holds for i = n.

Since

<z_n, z_n> = <$z_{n-1} + n^{-1/2}y_n$, $z_{n-1} + n^{-1/2}y_n$>

= <z_{n-1}, z_{n-1}> + <z_{n-1}, $n^{-1/2}y_n$> + $n^{-1/2}$<y_n, z_{n-1}> + 1/n <y_n, y_n>

and < z_{n-1}, y_n> ≥ 0, the assertion follows.

Now, for all t_1 we have

$$< x_{t_1}, x_{t_1} > \frac{1}{n_0}$$

and this gives

$$< z_n, z \; > \frac{1}{n_0}\left(\sum_{i=1}^{0} \frac{1}{i}\right)$$

which implies that the norm of z_n tends to ∞ and this gives that the norm of z is ∞. This contradiction proves the assertion.

Thus we are led to the following problem: What are the classes of Banach spaces which possess an inner product satisfying the axioms 1 through 4? For the treatment of this problem, we recall some notations.

Let X be a complex Banach space and X* its dual, Q is the closed unit ball with the weak topology. We show further that this is a compact Hausdorff space and from the extension theorems it follows that X is isomorphic and isometric with a closed subspace of C(Ω) with the norm

$f \to \| f \| = \sup\{|f(t)| \mid t \in \Omega\}$.

Suppose now that there exists a Radon measure μ whose support is Ω (the support of a Radon measure is the complement of the largest open set on which y vanishes). Clearly the support of u is equal to the support of |μ|, the total variation of μ.

Thus we can restrict our consideration to the case of positive μ Radon measures.

In this case, the formula

$$< x, y >< f_x, g_y >= \int f_x(t) q_y(t) d\mu(t)$$

defines an inner product on X, and of course f_x and q_y denote the images of x and y in C(Ω).

We remark that all the properties of the inner product are clear except possibly 1.

Let x ∈ C(Ω) and suppose that

$$\int_{\Omega} |x(t)|^2 \, d\mu(t) = 0$$

If for some $t_0 \in \Omega$, $x(t_0)) \neq 0$,

then from the continuity it follows that for some neighborhood V of t_0, x(t) ≠ 0 for all t ∈ V, and further there exists $\varepsilon_0 > 0$ such that

$|x(t)| \geq \varepsilon_0$ for t ∈ V.

In this case we have

$$\int_{\Omega} |x(t)|^2 \, d\mu \geq \int_{V} |x(t)|^2 \, d\mu \geq \varepsilon_0^2 \mu(V)$$

This contradicts the fact that the support of μ is Ω. This proves our assertion.

Example 3. Let Ω be a compact Hausdorff space and we suppose that it is separable, i.e., it contains a dense countable subset, say (t_i).

Let {a_i} be a sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1.$ In this case

$$\mu(f)\sum_{i=1}^{\infty} a_i f(t_i)$$

defines a Radon measure with the support equal to Ω.

Following Murphy and West (1972) we define the (ccc).

Definition 4. Let Ω be a topological space. We say that Ω satisfies the countable chain condition (ccc) if every disjoint family of nonempty sets in Ω is countable.

Remark 5. If Ω is separable, then it satisfies the (ccc).

Theorem 6. Let be a compact Hausdorff space. Then a necessary condition that supports a Radon measure is that it satisfy the (ccc).

Proof. Suppose that we have a Radon measure and

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