Pseudo quasi-conformal curvature tensor on K-contact manifold

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Abstract: The object of the present paper is to characterize K-contact manifold satisfying certain curvature conditions on the pseudo quasi-conformal curvature tensor.

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1. Introduction :

Let (M^n, g) (n=2m+1) be a contact Riemannian manifold. Let ϕ, ξ and η be tensor fields of the type (1,1), (1,0) and (0,1) respectively. If a contravariant vector field ξ is a killing vector field, then M^n is called a K-contact manifold Blair (1976) and Sasaki (1965). K-contact manifolds have been studied by many authors such a Tanno (1964, 1966, 1967), Sasaki (1967), Hatakeyama, Ogawa and Tanno (1963), Chaki and Ghosh (1972), Zhen (1992) and Zhen, Cabrerizo, Fernandez and Tripathi (2009), De and Ghosh (2009), Ghosh (2010), Tripathi and Gupta (2011) and many others. Further K-contact and Sasakian manifold with conservative concircular tensor, projective curvature tensor, quasi-conformal curvature tensor and E-curvature tensor studied by De and Tarafdar (1992), De and Ghosh (1993), De and Shaikh (1997), Prasad and Verma (2004) and many others.

Recently, the notion of the pseudo quasi-conformal curvature tensor τ on a Riemannian manifold was given by Shaikh and Janna in 2005. According to them a pseudo quasi-conformal curvature tensor was given by

$$\tau(X,Y)Z = (a+b)R(X,Y)Z + \left[c - \frac{b}{n-1}\right][S(Y,Z)X - S(X,Z)Y] + c[g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n(n-1)}[a + 2(n-1)c][g(Y,Z)X - g(X,Z)Y], \quad (1.1)$$

where a and b are constant and R, S, Q and r are the Riemannian curvature tensor of the type (1.3), the Ricci tensor of the type (0,2), the Ricci operator defined by S(X, Y) = g(QX, Y) and r is the scalar curvature of the manifold respectively.

The paper is organized as follows: Section 2 contains some preliminaries. In section 3, we prove that a pseudo quasi-conformally flat manifold is an η-Einstein manifold. Section 4 deals with the study of a K-contact manifold satisfying divτ = 0. Section 5 is devoted to the study of ξ pseudo quasi-conformally flat K-contact manifold.

2. Preliminaries :

Let M^n be an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η, g) consisting of a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and Riemannian metric g. Then,

$$\phi^2X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

From (2.1) and (2.2), we get

$$g(\phi X, Y) + g(X \phi Y) = 0 \text{ and } \qquad g(X, \xi) = \eta(X),$$

where X and Y arbitrary vector fields.

A contact metric manifold is Sasakian manifold if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.3}$$

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three June and Kim (1994). K-contact metric manifold are not to well known, because there is no such a simple expression for the curvature tensor as in the case of Sasakian manifold.

Besides the above relations on K-contact manifold the following relations hold (Blair, 1976; Sasaki, 1963, 1965, 1967 and Mishra, 1984),

$$D_X\xi = -\phi X, \tag{2.4}$$

$$g(R(\xi, X)Y\xi = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{2.6}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.7}$$

$$S(\xi, \xi) = (n - 1), \tag{2.8}$$

$$(D_X\phi) = R(\xi, X)Y, \tag{2.9}$$

where D is Riemannian connection.

Further since ξ is a killing vector field, S and r remains invariant under it, i.e.,

$$L_\xi S = 0 \text{ and } L_\xi r = 0, \tag{2.10}$$

where L denotes the Lie-derivation.

Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form S(X, Y) = λg(X, Y), where λ is a constant and η-Einstein if the Ricci tensor S is of the form S(X, Y) = Ag(X, Y) + B $\eta\otimes\eta$, where A and B are smooth functions on M. It is well known that in a K-contact manifold a and b are constants June and Kim (1994). Also it is known that every manifold of constant curvature is an Einstein manifold is Sasakian Boyer and Galicki (2001).

3. Pseudo quasi-conformally flat K-contact manifolds :

In this section we consider the pseudo quasi-conformally flat K-contact manifold. If a K-contact manifold (M^n, ϕ,ξ,η, g) is pseudo quasi-conformally flat, then from (1.1), we get,

$$(a + b)R(X,Y)Z = \left[c - \frac{b}{n-1}\right][S(X,Z)Y - S(Y,Z)X] + c[g(X,Z)QY - g(Y,Z)QX] - \frac{r}{n(n-1)}.[a + 2(n-1)c][g(X,Z)Y - g(Y,Z)X]. \quad (3.1)$$

Form (3.1), we have

$$(a + b)'R(X,Y,Z,W) = \left[c - \frac{b}{n-1}\right][S(X,Z)g(Y,W) - S(Y,Z)g(X,W)] + c[g(X,Z)S(Y,W) - g(Y,Z)S(X,W)] - \frac{r}{n(n-1)}.[a + 2(n-1)c][g(X,Z)g(Y,W) - g(Y,Z)g(X,W)], \quad (3.2)$$

where, $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Putting ξ for X and Z in (3.2), we find

$$(a + b)\,'R(\xi,Y,\xi,W) = \left[c - \frac{b}{n-1}\right][S(\xi,\xi)g(Y,W) - S(Y,\xi)g(\xi,W)] + c[g(\xi,\xi)S(Y,W) - g(Y,\xi)S(\xi,W)] - \frac{r}{n(n-1)}.[a + 2(n-1)c][g(\xi,\xi)g(Y,W) - g(Y,\xi)g(\xi,W)]. \quad (3.3)$$

Now using (2,1), (2,2), (2.3) and (2.7) in (3.3), we get,

$$S(Y,W) = Ag(Y,W) + B\eta(Y)\eta(W), \quad (3.4)$$

where A and B are given by,

$$A = -\frac{a}{c} + \frac{r}{nc}\left(\frac{a}{n-1} + 2c\right) - (n-1),$$

and (3.5)

$$B = -\frac{a}{c} + \frac{r}{nc}\left(\frac{a}{n-1} + 2c\right) - 2(n-1).$$

In view of relation (3.4), we state the following theorem :

Theorem 3.1 : A pseudo quasi conformally flat K-contact manifold is η- Einstein manifold.

Putting e_i for Y and W, where e_i is an orthonormal basis of the tangent space at each point of the manifold in (3.4), and taking summation over i, $1 \le i \le n$, we get,

$$r = An + B. \quad (3.6)$$

Now with the help of (3.5) and (3.6), we get,

$$[a + (n-2)c][r - n(n-1)] = 0 \Rightarrow r = n(n-1), a + (n-2)c \neq 0. \quad (3.7)$$

Hence, we can state the following theorem :

Theorem 3.2 : In pseudo quasi conformally-flat K-contact manifold, a scalar curvature $r = n(n - 1)$, provided $a + (n - 2)c \neq 0$.

From (3.4), (3.5) and (3.7), we get

$$S(Y,W) = (n-1)g(Y,W). \quad (3.8)$$

In view of (3.8), we can say that M^n is an Einstein manifold.

Putting (3.7) and (3.8) in (3.2), we find,

$$(a + b)['R(X,Y,Z,W) - g(Y,Z)g(X,W) + g(Y,W)g(X,Z)] = 0,$$

$$\Rightarrow 'R(X,Y,Z,W) = g(Y,Z)g(X,W) - g(Y,W)g(X,Z) = 0, (a + b) \neq 0. \qquad (3.9)$$

The relation (3.9) shows that manifold is constant curvature +1, provided a + b ≠ 0. This leads to the following theorem :

Theorem 3.3 : A pseudo quasi-conformally-flat K-contact manifold is a constant curvature +1, provided a + b ≠ 0.

Since the manifold under consideration is a constant curvature. Therefore by this result we get the following corollary:

Corollary (3.1) : A pseudo quasi conformally-flat K-contact manifold is a Sasakian manifold.

4. K-contact manifold satisfying Div(τ) = 0 :

This section deals with a K-contact Riemannian manifold satisfying :

$$\operatorname{div}(\tau) = 0, \qquad (4.1)$$

where div denotes the divergence of the pseudo quasi conformal curvature tensor.

Differentiating (4.1) covariantly along U, we get

$$(D_U r)(X,Y)Z = (a + b)(D_U R)(X,Y)Z + \left[c - \frac{b}{n-1}\right].[(D_U S)(Y,Z)X - (D_U S)(X,Z)Y] + c[g(Y,Z)X - (D_U S)(X,Z)Y] - \frac{D_U r}{n}\left[\frac{a}{n-1} + 2c\right][g(Y,Z)X - g(X,Z)Y]. \qquad (4.2)$$

Contraction of (4.2) gives

$$(\operatorname{div}\tau)(X,Y)Z = \left[a + b + c - \frac{b}{n-1}\right][(D_X S)(Y,Z) - (D_Y S)(X,Z)] + \frac{c(n-1)(n-4) - 2a}{2n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \qquad (4.3)$$

In view of (4.1) and (4.3), we get,

$$\left[a + b + c - \frac{b}{n-1}\right][(D_X S)(Y,Z) - (D_Y S)(X,Z)] + \frac{c(n-1)(n-4) - 2a}{2n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0, \qquad (4.4)$$

From (2.9), we respectively,

$$(D_\xi S)(Y,Z) = -S(DY\xi, Z) - S(Y, DZ\xi) \quad \text{and } dr(\xi) = 0. \qquad (4.5)$$

Putting ξ for X in (4.4) and using (4.5), we find,

$$\left[a + b + c - \frac{b}{n-1}\right][(D_Y\xi, Z) - (D_Z\xi, Y) + (D_Y S)(\xi, Z)] + \frac{c(n-1)(n-4) - 2a}{2n(n-1)}[\eta(Z)dr(Y)] = 0. \qquad (4.6)$$

From (2.7), we get

$$(D_Y S)(\xi, Z) = (n-1)g(D_Y\eta)(Z) - S(D_Y\xi, Z). \qquad (4.7)$$

Again using the relation $(D_Y\eta)(Z) = g((D_Y\xi, Z)$ in (4.7), we get

$$(D_X S)(\xi, Z) = (n-1)g(D_X\xi, Z) - S(D_Y\xi, Z). \quad (4.8)$$

Using (4.8) in (4.6), we find,

$$\left[a + b + c - \frac{b}{n-1}\right][S(D_Z\xi, Y) + (n-1)g(D_Y\xi, Z)] + \frac{c(n-1)(n-4) - 2a}{2n(n-1)}[\eta(Z)dr(Y)] = 0. \quad (4.9)$$

In view of (2.4), (4.9) can be put as

$$\left[a + b + c - \frac{b}{n-1}\right][S(\phi Z, Y) + (n-1)g(\phi Y, Z)] + \frac{c(n-1)(n-4) - 2a}{2n(n-1)}.\eta(Z)dr(Y) = 0. \quad (4.10)$$

Replacing Z by ϕZ in (4.10) and using (2.1), we have,

$$S(Y, Z) = (n-1)g(Y, Z), \text{ provided } (n-1)(a + c) - (n-2)b \neq 0. \quad (4.11)$$

Hence, we can state the following theorem:

Theorem 4.1 : A K-contact manifold with divergence free pseudo quasi conformal curvature tensor is an Einstein manifold provided $(n – 1) (a + c) – (n – 2)b \neq 0$.

5. ξ–pseudo quasi conformally flat K-contact manifolds :

ξ-conformally flat K-contact manifold, ξ-quasi conformally flat K-contact manifolds, ξ-conharmonically falt K-contact manifolds have been studied by Zhen, cabrerizo, fernanadez and fernanadez (1997), De and Ghosh (2009) and Dwivedi and Kim (2011) respectively. Hence we study ξ-pseudo quasi conformally flat K-contact manifold.

Definition (5.1) : A K-contact manifold is said to be ξ-pseudo pseudo quasi conformally flat if $\tau\tau(X, Y)\xi = 0$, Zhen, Gabrerizo, fernanadez Fernandez, (1997).

Let us assume that the manifold M^n is ξ-pseudo quasi conformally flat. Then using $\tau(X, Y)\xi = 0$ in (1.1), we find

$$(a + b)R(X, Y)\xi + \left[c - \frac{b}{n-1)}\right][S(Y, \xi)X - S(X, \xi)Y + c[g(Y, \xi)QX - g(X, \xi)QY] - \frac{r}{n(n-1)}[a + 2(n-1)c].[g(Y, \xi)X - g(X, \xi)Y] = 0. \quad (5.1)$$

Putting ξ for X in (5.1) and using (2.6) and (2.1), we get,

$$(a + b)\left[-Y + \eta(Y)\xi + \left[c - \frac{b}{n-1)}\right](n-1)\right][\eta(Y)\xi - Y] + c[(n-1)\eta(Y)\xi - QY] - \frac{r}{n(n-1)}[a + 2(n-1)c].[\eta(Y)\xi -] = 0, \quad (5.2)$$

which on simplification gives,

$$S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W), \quad (5.3)$$

where A and B are given by

$$A = -\frac{a}{c} + \frac{r}{nc}\left(\frac{a}{n-1} + 2c\right) - (n-1),$$

and (5.4)

$$B = -\frac{a}{c} - \frac{r}{nc}\left(\frac{a}{n-1} + 2c\right) - 2(n+1).$$

Hence, in view of (5.2), we have the following theorem:

Theorem 5.1 : A ξ-pseudo quasi conformally flat K-contact manifold is an η-Einstein manifold.

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