A Review of Cut Points in Topological Connected Space IR

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ABSTRACT:

The Topological Space IR (Real Line) has the followings properties:

a. It is connected but the removal of any one of its points leaves it disconnected.

b. It is metrizable.

c. Its topology can be generated by a linear ordering.

The topological space that satisfy condition a, is called cut point space. Formally we define a cut point topological. Let X be a non-empty connected topological space. A point x in X is said to be a cut point of X if X/{x} is a disconnected subset of X. A non-empty connected topological space X is said to be a cut point space if every x in X is a cut point of X. A point x of a space X is called a cut point if there exists a separation of X-{x}.

KEYWORDS: Topological Space, Metrizable, Cut Points.

INTRODUCTION:

The concept of cut points plays a very important role in topological spaces. The real line IR and the khalimsky line Z are the example of cut point spaces. Both are connected topological spaces but removal of any one of the points of IR or Z leaves the space disconnected, such points are called cut-points. In section (2) we will study topological properties of a cut points spaces. The important result of this section is that cut point topological space has infinite set of closed points. Also we will prove that cut points space is non compact. In section (3) we will define irreducible cut point space and prove that it is homeomorphic to the ‘Khalimsky line’.

Topological Properties of Cut Point Spaces:

**Theorem 1:** Let X be a connected topological and let x be a cut point of X such that X/{x}=A/B then {x} is open or closed. If {x} is open then A and B are closed. If {x} is closed then A and B are open.

**Proof:** Here {x} is a cut points of connected topological space X. so by definition X|{x} is disconnected. We have X/{x}=A/B=> X|{x}=AUB where A∩B=ϕ and Ā∩B= ϕ. Consider X/{x} as a sub space of X, then A is both open and closed in X|{x} similarly B. So if A is open in X|{x} then ∃ open subset G of X such that A= G∩X/{x} (Type of open subset of sub space) Since A=G∩X/{x}] and G⊂X=>A=G/ {x}......(2.1) Now if A is closed in X/{x}
then ∃ closed subset F of X such that \( A=F\cap (X/{x}) \) again \( F\subseteq X=\{x\}=F/\{x\}=F/\{x\} \)....(2.2) From (2.1) and (2.2) we have: \( A=G/\{x\}=F/\{x\} \). Now if \( G=F \) then \( G \) and \( F \) both are open and closed subset of \( X \). So \( A=G \) and \( G=F \) then \( \{x\} \) is closed and \( \{x\} \) is a singleton then \( X \) has at least two non cut points. Since \( X \) has only one non cut point and \( B \) contains it, \( X \) is a cut points space. 

**Examples of cut-points spaces:**

**Example:** Let \( X_1=\{(x,y) \in \mathbb{R}^2 : x \leq 0 \text{ and } |y|=1\} \). Let \( X_2=\{(x,y) \in \mathbb{R}^2 : x>0 \text{ and } y=\sin \frac{1}{x}\} \). Define \( X=X_1\cup X_2 \) then \( X \) is a cut point space.

**Example:** (The Khalimsky Line). Let \( Z=\text{set of integers and let } \beta=\{\{2i-1,2i,2i+1\}; i \in \mathbb{Z}\} \cup \{(2i+1) ; i \in \mathbb{Z}\} \). Then \( \beta \) is a base for topology on \( Z \). The set \( Z \) with this topology is a cut point space and is called the Khalimsky line:

**Notation:** Connected ordered topological space -> COTS

**Theorem 2:** Let \( X \) be a cut point space. Then set of closed points of \( X \) is infinite.

**Proof:** suppose if possible the set \( A=(x_1,x_2,...,x_n) \) of distinct closed points of \( X \) is finite. Since \( X \) is a cut points space so considered \( X \) as a set of all cut points of \( X \) put \( C_0=X \). Now as a connected subset of \( X \) which is not singleton \( C_0 \) contains at least one closed point say \( x_1 \) (by rearrangement) since \( \{x_1\} \) is a cut points of \( X \). By definition \( \exists \) open subsets say \( C_1 \) and \( D_1 \) of \( X \) such that \( X/\{x_1\}=C_1/D_1 \). Now suppose that the distinct closed points \( x_1,x_2,...,x_n \) in \( X \) and open subsets \( C_1,C_2,...,C_n,D_1,D_2,...,D_n \) of \( X \) are chosen such that \( X/\{x_1\}=C_1/D_1 \) \( \text{ and } x_1 \in C_1 \) and \( C_i\subseteq C_i \) for every \( i \), \( 1 \leq i \leq n \). Now \( X/\{x_n\}=C_n/D_n \) and \( C_n \) contains one closed point say \( x_{n+1} \). We will prove that the closed point \( x_{n+1} \) is different from \( x_1,x_2,...,x_n \). Now \( x_{n+1} \) is a cut point of \( X \) so \( \exists \) open subsets \( C_{n+1} \) and \( D_{n+1} \) of \( X \) such that \( X/\{x_{n+1}\}=C_{n+1}/D_{n+1} \). By interchanging \( C_{n+1} \) and \( D_{n+1} \), if necessary; we may assume that \( x_n \in D_{n+1} \). Now \( X/\{x_n\}=C_n/D_n \) and \( X/\{x_{n+1}\}=C_{n+1}/D_{n+1} \). We have \( x_{n+1} \in C_n \) and \( x_n \in D_n \). So \( C_n \subseteq C_{n+1} \). Also \( x_i \in C_i \) and \( C_i \subseteq C_i \) for every \( i \); \( 1 \leq i \leq n \) \( \Rightarrow x_i \in C_i \) for every \( i \); \( 1 \leq i \leq n \). But \( x_{n+1} \in C_n \Rightarrow x_{n+1} \) is different from \( x_1,x_2,...,x_n \). So our assumption is wrong. I.e. the set \( A \) of all closed points of \( X \) is not finite. It is infinite set.

**Theorem 3:** let \( X \) be a cut points space. Then \( X \) is non-compact.

**Proof:** - To prove the result, we will prove that “if \( X \) is a compact connected topological space which is not singleton then \( X \) has at least two non-cut points. I.e. \( X \) is not a cut point space. Suppose if possible, \( X \) has at most one non cut point. Let \( x_0 \) be a cut point of \( X \) \( \Rightarrow \exists \) open subsets say \( A_0 \) and \( B_0 \) of \( X \) such that \( X/\{x_0\}=A_0/B_0 \). We assume that \( B_0 \) contains that one non cut point. Since \( X \) has only one non cut point and \( B_0 \) contains it \( \Rightarrow \) every point of \( A_0 \) is a cut point of \( X \). So \( A_0 \) contains at least one closed point say \( X \). Again \( x \) is a cut point of \( X \). so \( \exists \) A and B open subsets of X such that \( X/\{x\}=A/B \) also \( X/\{x_0\}=A_0/B_0 \) and \( x_0 \in B_0 \Rightarrow A \subseteq A_0 \). Now define \( S=(U:U \subseteq X \text{ is open, } U \not\subseteq B; \bar{U}/U \text{ is singleton and } \bar{U} \neq X) \). Since \( B \) is open \( \exists \) B and \( \beta=B\cup U(x) \) i.e. \( B/\beta \) is singleton; also \( B=\alpha \times S. \) Define partial order in S by if \( U_\alpha,U_\beta \) then \( U_\alpha \leq U_\beta \) if \( U_\alpha =U_\beta \text{ or } U_\alpha \leq U_\beta \). Clearly \( (S, \leq) \) is
a partially ordered set; by Hausdorff maximal principle \( \exists \) a maximal chain say \( \zeta \in S \). let \( U_0 \in S \) and \( \tilde{U}_0 = U_0 \cup \{x_0\} \). Since \( X/(x_0) = U_0/(X/\tilde{U}_0) \). Again \( \exists \) a closed point say \( y \in X/\tilde{U}_0 \subseteq A \). again \( y \) is a cut point so \( \exists \) open subset \( C \) and \( D \) of \( X \) such that \( X/y = C/D \). Since \( \tilde{U}_0 = U_0 \cup \{x_0\} \) so \( \tilde{U}_0 \) is connected. Since \( X/y = C/D \) so either \( \tilde{U}_0 \subseteq C \) or \( \tilde{U}_0 \subseteq D \). 

Since \( U_0 \) was arbitrary in \( S \); \( S \) does not have a maximal element. Thus \( \bigcup_{\nu \in \zeta} U_{\nu} = \bigcup_{\nu \in \zeta} \tilde{U}_{\nu} \) 

say \( V = \bigcup_{\nu \in \zeta} U_{\nu} \) 

\( \nu \in \zeta \) 

since \( \tilde{U} \) is connected for every \( U \in S \); \( V \) is also connected. We claim that \( V = X \). suppose if possible \( X/V \) is a non-empty closed subset of \( X \). Since \( X/V \subseteq A \) and \( A \subseteq A_0 \) i.e. \( X/V \subseteq A \rightarrow \) every point of \( X/V \) is a cut point of \( X \). so by theorem 2.1 \( X/V \) is either open or closed. Since \( X/V \) is closed so \( \exists \) a closed cut point say \( x^1 \in X/V \). Since \( X/V \subseteq X \) so \( x^1 \) is also a cut point of \( X \) so \( \exists \) open subset \( G \) and \( H \) of \( X \) such that \( X/\{x^1\} = G/H \). Since \( V \) is connected and either \( V \subseteq G \) or \( V \subseteq H \). Assume \( V \subseteq G \) since \( G \) is arbitrary any \( U \in C \). Since \( \zeta \) does not have a maximal element \( G \subseteq \zeta \). This contradicts the maximality of the chain \( L \). 

Hence \( V = X \). So \( X = U \cup_{\nu \in \zeta} U_{\nu} \) 

| i.e. \( X \) has an infinite open covers. Since \( \zeta \) is a chain without maximal elements so \( X \) has no finite sub covers => \( X \) is non-compact. Which is contradiction to the given condition that \( X \) is compact. So our supposition is wrong. \( X \) has at least two non cut point i.e. if \( X \) is compact then \( X \) is not a cut point space. Every cut point space is non-compact. 

Section: Definition:- A cut point space is said to be an irreducible cut point space if no proper subset of it ( with the subspace topology) is a cut point space. 

**Lemma 1** :- Let \( X \) be a cut point space. Let \( x \in X \). Let \( X/\{x\} = A/B \). if \( A \) is not connected then \( AU\{x\} \) is a cut point space. 

**Proof** - Put \( Y = A \cup \{x\} \). clearly \( x \) is a cut point of \( Y \). let \( y \in A \) we will prove that \( y \) is a cut point of \( A \). we have \( X/\{x\} = A/B \) so \( X/y = (Y/y)U(BU\{x\}) \) is not connected also \( x \in Y = A \cup \{x\} \) and \( x \in BU\{x\} \) so \( x \in (y/\{y\}) \cap (BU\{x\}) \). So either \( Y/y \) or \( BU\{x\} \) is disconnected. But \( BU\{x\} \) is connected. So \( Y/y \) is disconnected i.e. \( y \) is cut point of \( Y = AU\{x\} \). Since \( y \) was arbitrary. So every point of \( AU\{x\} \) is cut point => \( AU\{x\} \) is a cut point space. 

**Theorem 2**:- If \( X \) is an irreducible cut point space, then for every \( x \in X \), \( X/\{x\} \) has exactly two components. 

**Proof** - Let \( X/\{x\} = A/B \) since \( X \) is irreducible, so \( AU\{x\} \) and \( BU\{x\} \) are not cut point spaces. So by above lemma; \( A \) and \( B \) both are connected. 

**Lemma 3**:- Let \( X \) be an irreducible cut points space, let \( x \in X \) and Let \( X/\{x\} = A/B \). Then there are exactly two points \( y \in A \) and \( z \in B \) such that \( \{x, y\} \) and \( \{x, z\} \) are connected. Futher more if \( x \) is closed then \( y \) and \( z \) are open and if \( x \) is open then \( y \) and \( z \) are closed.
Proof:- Since X is irreducible cut point space so by theorem 3.2 $X/\{x\}$ has exactly two components now we have $X/\{x\}=A/B$ so both A and B are connected. Also X is irreducible cut point space so by definition; A is not a cut point space (as proper subset). \(\exists \ y \in A \) such that $y$ is not a cut point in A i.e. $A/\{y\}$ is connected. To prove that $(x,y)$ is connected; we will prove that if $y^1 \in A$ such that $(x,y^1)$ is connected then $y=y^1$.

We have $X/\{x\}=A/B$ where $x$ is a cut point of X. So $BU\{x\}$ is connected. Also we can put $X/\{y\}=(A/\{y\})U(BU\{x\})$.

Since $A/\{y\}$ is connected $BU\{x\}$ is connected also $(x,y^1)$ is connected \(\Rightarrow X/\{y\}\) is connected which is a contradiction to the fact that $y$ is cut point of X (X is cut point space so every element. as cut point). Now consider the two cases:

Case 1:- Let $x$ is closed. Then by theorem (2.1) A is open. But $AU\{x\}$ is closed. So $x$ is limit point of A. Also we have $X/\{y\}=(A/\{y\})U(BU\{x\})$. Since $x$ is a limit point of A so $(A/\{y\})\cap(BU\{x\})=\emptyset$. So $X/\{y\}$ is not disconnected which is a contradiction to the fact that $y$ is cut point of X. so $x$ is not a limit point of $A=\Rightarrow X$ is limit point of $\{y\}$ so $(x,y)$ is connected.

Case 2:- $x$ is open \(\Rightarrow A\) is closed but $AU\{x\}$ is open. Since A is closed \(\Rightarrow A\) is not open \(\Rightarrow \exists \ y' \in A\) such that $A$ is not neighborhood of $y'$. Since $y'$ is interior point of $AU\{x\} \Rightarrow y'$ is limit point of $\{x\}=\Rightarrow (x,y')$ is connected, from above $y'=y$ so $(x,y)$ is connected. Similarly we can prove that $\exists$ unique $z \in B$ such that $(x,z)$ is connected. Also by theorem (2.1) if $x$ is closed then $y$ and $z$ are open and if $x$ is open then $y$ and $z$ are closed.

Theorem:- A topological space $X$ is an irreducible cut point space if and only if $X$ is homeomorphic to the ‘Khalimsky line’.

Proof:- Clearly the ‘Khalimsky line’ is an irreducible cut point space. Let $X$ be an irreducible cut point space. Let $x_0$ be any closed point in $X$; Let $X/\{x_0\}=A_0/B_0$. Then by lemma 3.3 \(\exists\ x_j \in A_0\) and $x_1 \in B_0$ such that $\{x_0,x_1\}$ and $\{x_0,x_1\}$ are connected. Define $Y_1=\{x_1, x_0x_1\}$. Let $A_1$ be the component of $X/\{x_1\}$ that contains $x_0$ and let $B_1$ be the other components of $X/\{x_1\}$. Let $B_1$ be the components of $X/\{x_1\}$ that contains $x_0$ and let $A_1$ be the other components of $X/\{x_1\}$. Assume that for an arbitrary positive integer $n$ the subset.

$\{x_i : i \in Z \text{ and } -n \leq i \leq n\}$ of $X$ is chosen such that for each $i$ and $j$ which satisfy $-n \leq i, j \leq n$ and $|i-j|=1$, $(x_i, x_j)$ is connected. Moreover assume that for each non-zero $i$, $-n \leq i \leq n$, the components $A_i$ and $B_i$ of $X/\{x_i\}$ chosen such that $x_0 \in A_i$; if $i$ is positive and $x_0 \in B_i$, if $i$ is negative since $Y_n/\{x_n\}= U(x_i, x_i)$ is connected, it is a subset of $A_{-n}$ or $B_{-n}$, and since $x_0 \notin A_{-n}$ and $Y_n/\{x_n\} \subset B_{-n}$. By Lemma 4.4 there is a unique point $x_{-n+1}$ in $A_{-n}$ such that $\{x_{-n}, x_{-n+1}\}$ is connected. Since $(Y_n U \{x_{-n+1}\})/\{x_n\}=U(x_i, x_i)$ is connected, it is a subset of $A_n$ or $B_n$, and since $x_0 \notin E B_n$, $(Y_n U \{x_{-n+1}\})/\{x_n\}$ is connected. It is a subset of $A_n$ or $B_n$ and since $x_0 \notin E B_n$, $(Y_n U \{x_{-n+1}\})/\{x_n\}$ is connected. It is a subset of $A_{-n}$ or $B_{-n}$ such that $\{x_{-n}, x_{-n+1}\}$ is connected. Thus we obtain a subset $Y_{n+1}=\{x_i : i \in Z \text{ and } -(n+1) \leq i \leq n+1\}$ of $X$ (with $2n+3$ points) such that for each $i$ and $j$ which satisfy $-(n+1) \leq i, j \leq n+1$ and $|i-j|=1$, $(x_i, x_j)$ is connected. To complete the induction step, we define the subsets $A_{n-1}, B_{n-1}, A_{n+1}, B_{n+1}$ of $X$ such that $X/\{x_{n-1}\}=A_{n-1}/B_{n-1}$, $X/\{x_{n+1}\}=A_{n+1}/B_{n+1}$, $x_0 \notin E B_{n-1}$ and $x_0 \notin E A_{n+1}$. Put $Y = U Y_n=\{x_i : i \in Z\}$.

It can be easily seen that for each integer $i$, $Y \cap A_i = \{x_j : j < i\}$ and $Y \cap B_i = \{x_j : j > i\}$. Since $x_0$ is closed, (by iterated application of Lemma 4.4) $x_n$ is closed if $n$ is even, and $x_n$ is open if $n$ is odd. Clearly, for each $i \in Z$, the smallest open neighborhood of $x_{2i+1}$ in $Y$ is $\{x_{2i+1}\}$. Since for each $i \in Z$, $x_{2i+1}$ is a limit point of $\{x_{2i}\}$ and
\{x_{2i+1}\}, every open neighborhood of \(x_{2i}\) in X (and hence in Y) contains \(x_{2i-1}\) and \(x_{2i+1}\). On the other hand, since \(x_{2i-2}\) and \(x_{2i+2}\) are closed, \(B_{2i-2}\) and \(A_{2i+2}\) are open in X. Thus \(\{x_{2i-1}, x_{2i}, x_{2i+1}\} = (Y \setminus B_{2i-2}) \setminus (Y \cap A_{2i+2})\) is the smallest open neighborhood of \(x_{2i}\) in Y. Hence \(B^1 = \{\{x_{2i-1}, x_{2i}, x_{2i+1}\} : i \in \mathbb{Z}\} \cup \{\{x_{2i+1}\} : i \in \mathbb{Z}\}\) is a base for the topology of Y.

Comparing this base with the base of the Khalimsky line in Example 2.5, we see that Y is homeomorphic to the Khalimsky line.

References: