SOME SPECIAL PYTHAGOREAN TRIANGLES

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Abstract : Special Pythagorean Triangles, in terms of perimeter to be a sum of three squares, with consecutive sides are obtained. A few interesting results are observed.

Key Words - Pythagorean Triangles, sum of three sides is a square.
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I. INTRODUCTION

In the Pythagorean Mathematics, the determination of classifying all Pythagorean triangles wherein each of which the perimeter can always be represented as the sum of three distinct squares by Gopalan and Devibala **[1]**, motivates to examine whether there exists any Pythagorean Triangle, with two consecutive sides and sum of legs to be a square. It is towards this end an attempt is made to find patterns of Special Pythagorean Triangles with two consecutive sides and perimeter as sum of three squares.

II. METHOD OF ANALYSIS

The primitive solutions of the Pythagorean Equation

$$\mathbf{X^2 + Y^2 = Z^2} \quad (1)$$

is given by **[2]**

$$X = m^2 - n^2,\ Y = 2mn,\ Z = m^2 + n^2 \quad (2)$$

where $\boldsymbol{m, n} \in I$ such that $\boldsymbol{m > n > 0}$ and $(m, n) = 1$ with one of them is odd other even.

2.1 Sum of two legs is a square:

$$X + Y = \beta^2 \quad (3)$$

Then solutions of (3) is given by **(4)**

$m = 2p^2 + q^2 - 2pq,$

$n = 2pq,$

$\beta = 2p^2 - q^2$, p and q are positive integers.

(2) and (4) give

$X = (2p^2 + q^2)(2p^2 + q^2 - 4pq)$

$Y = 4pq(2p^2 + q^2) - 8p^2q^2$

$$Z = (2p^2 + q^2 - 2pq)^2 + (2pq)^2 \quad (5)$$

2.2 Hypotenuse and one leg are consecutive:

In such cases, m = n + 1

$\Rightarrow 2p^2 + q^2 - 4pq = 1$

$\Rightarrow q = 2p \pm \sqrt{(1 + 2p^2)}$ **(6)**

Applying (6) to (5), we get

$X = 1 + 4pq$

$Y = 4pq\,(1+ 2pq)$

$Z = 4pq\,(1+ 2pq) + 1$ (7)

2.3 Perimeter is sum of three squares:

From (2) and (3),

$X + Y + Z = \beta^2 + m^2 + n^2$ **(8)**

(2), **(4)**, **(6)** and **(7)** generates X and Y which satisfy **(1)** and **(3)** in correspondence with **(8)**.

Few examples are given in the table 1 below:

Table1

C□	**1**	**2**	**3**	**4**	**5**	**6**	**7**
R↓	**p**	**q**	**M**	**n**	**X**	**Y**	**Z**
1	2	1	5	4	9	40	41
2	2	**7**	29	28	57	1624	1625
3	12	**7**	169	168	337	56784	56785
4	12	**41**	985	984	1969	1938480	1938481
5	70	**41**	5741	5740	11481	65906680	65906681
6	70	**239**	33461	33460	66921	2239210120	2239210121
7	408	**239**	195025	195024	390049	76069111200	76069111201
8	408	**1393**	1136689	1136688	2273377	2584121492064	2584121492065
9	2378	**1393**	6625109	6625108	13250217	87784125273544	87784125273545
10	**?**	**8119**	**?**	**?**	**?**	**?**	**?**

C□	**8**	**9**	**10**
R↓	$\mathbf{X + Y = \beta^2}$	**β**	$\mathbf{X + Y + Z = \beta^2 + m^2 + n^2}$
1	$9 + 40 = 49 = 7^2$	**7**	$90 = 9 + 40 + 41$ $= 7^2 + 5^2 + 4^2$
2	$57 + 1624 = 41^2$	**41**	$3306 = 57 + 1624 + 1625$ $= 41^2 + 29^2 + 28^2$

3	$337 + 56784 = 57121 = 239^2$	**239**	$113906 = 337 + 56784 + 56785$ $=239^2 + 169^2 + 168^2$
C□	**8**	**9**	**10**
R↓	$\mathbf{X + Y = \beta^2}$	$\boldsymbol{\beta}$	$\mathbf{X + Y + Z = \beta^2 + m^2 + n^2}$
4	$1969 + 1938480 = 1940449$ $= 1393^2$	**1393**	$3878930 = 1393^2 + 985^2 + 984^2$
5	$11481 + 65906680$ $= 65918161 = 8119^2$	**8119**	$131824842 = 8119^2 + 5741^2 + 5740^2$
6	$66921 + 2239210120 = 2239277041$ $= 47321^2$	47321	$4478487162 = 47321^2 + 33461^2$ $+33460^2$
7	$390049 + 76069111200 = 76069501249$ $=275807^2$	275807	$152138612450 = 275807^2 +$ $195025^2 + 195024^2$
8	$2273377 + 2584121492064 =$ $2584123765441 = 1607521^2$	1607521	$5168245257506 = 1607521^2 +$ $1136689^2 + 1136688^2$
9	$87784138523761 = 13250217 +$ $87784125273544 = 9369319^2$	9369319	$175568263797306 = 9369319^2 +$ $6625109^2 + 6625108^2$
10	**?**	**?**	**?**

III OBSERVATIONS

We observe that

1. $X + Y + Z = 2 (1 + 4pq) (1 + 2pq)= 0(\text{mod } 2)$
2. The Pythagorean Triplets (X,Y,Z) given by (6) are all primitives as $Z = Y + 1$
3. $(Y + Z – X)^2 = 2(Y + Z)(Z – X)$
4. $(X + 2Y + Z)^2 = (Z – X)^2 + 4(X + Y)(Y + Z)$
5. $X + 2Y + Z \pm 2\{ (X + Y)(Z - Y)\}^{1/2} = 0(\text{mod}16)$ or $= 0(\text{mod}4)$
6. Except for $q = 1$, the values of β and the values of q are same with q repeated twice.
7. For every value of p, there are two values of q and vice-versa.

Thus, the question arises: whether q can be obtained from previously obtained β and hence p?

From (6) we get $2p^2 + q^2 - 4pq = 1$.

Putting $q = 8119$, we get, $p = 2378$ and $p = 13860$.

Solving for m, n, X, Y, Z and β when p = 2378 using software *Mathematica*, we get

m = 38613965, n = 38613964, X = 77227929, Y = 2982076508814520, Z = 2982076508814521 and

β = 54608393 which satisfy (7) (Table 2)

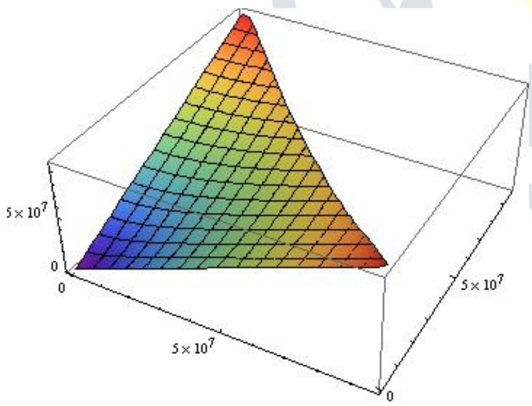
Table 2

C□	1	2	3	4	5	6	7
R↓	**p**	**q**	**M**	**n**	**X**	**Y**	**Z**
10	2378	**8119**	38613965	38613964	77227929	2982076508814520	2982076508814521

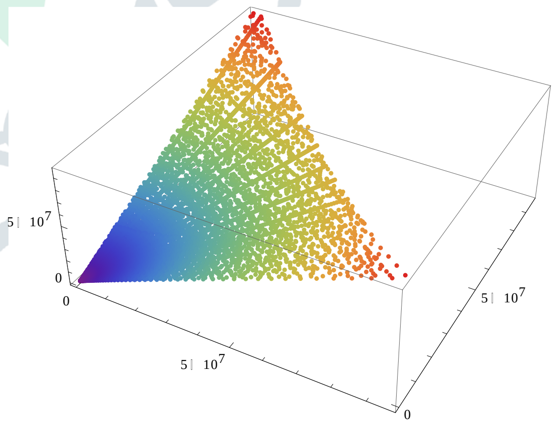
C□	8	9	10
R↓	**X + Y = β^2**	**β**	**X + Y + Z = $\beta^2 + m^2 + n^2$**
10	2982076586042449 = 77227929 + 2982076508814520 = 54608393^2	54608393	5964153094856970 = $54608393^2 + 38613965^2 + 38613964^2$

IV 3D PLOTS

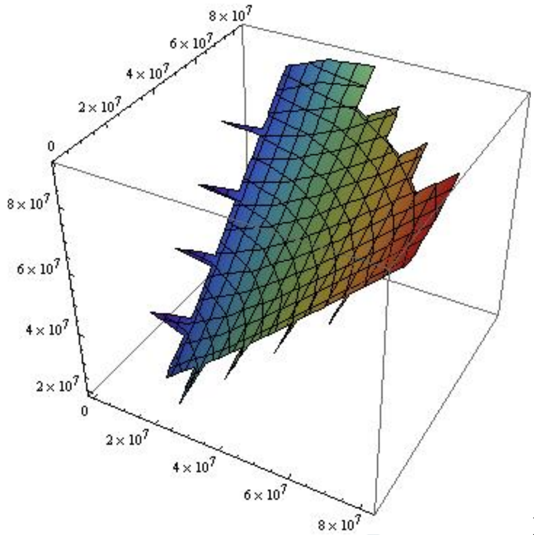
Solving the Diophantine equation $m^2 - n^2 + 2mn = \beta^2$ i.e., X +Y = β^2 with the help of ***Mathematica*** for for n < m; 0 < m < 10000; 0 < n < 10000; 0 < β < 10000, we get 9354 solutions of which ListPlot3D, ListPointPlot3D and L istSurfacePlot3D are given below:



ListPlot3D



ListPointPlot3D



ListSurfacePlot3D

For consecutive leg and hypotenuse and X +Y = β^2, solving the equation

$(n + 1)^2 - n^2 + 2(n + 1)n = \beta^2$ using the software ***Mathematica*** for the values $0 < n < 10^{10}$ and $0 < \beta < 10^{10}$ we get only 130 solutions!

In conclusion, one may attempt to find other patterns of Pythagorean Triangle which satisfy the conditions presented in the above problem.

References

[1] Gopalan, M.A.. Devibala.S. 2007.Special Pythagorean Triangles. The Mathematics Education, Vol. XLI(4): 293.
[2] Burton, David M. 2009. Elementary Number Theory. Tata MacGraw-Hill Edition. New Delhi: 248.