Multiparameter S-K-Mittag-Leffler Function

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Abstract : **In this paper we introduce Multiparameter S-K-Mittag-Leffler Function defined as,**

$$ {}_p^qK_{s,k}^{(\beta,\eta)_m}[z] = {}_p^qK_{s,k}^{(\beta,\eta)_m}[a_1,..,a_p;b_1,..,b_q,(\beta_1,\eta_1),..,(\beta_m,\eta_m);z], $$

$$ {}_p^qK_{s,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\ {}_s(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q}\ {}_s(b_r)_{n,k} \prod_{i=1}^{m}\ {}_s\Gamma_k(\eta_i n+\beta_i)}, $$

where $s,k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$.

Certain relations that exist between ${}_p^qK_{s,k}^{(\beta,\eta)_m}[z]$ **function and Riemann-Liouville fractional integral and derivatives have been evaluated. It has been shown that the fractional integration and differentation of** ${}_p^qK_{s,k}^{(\beta,\eta)_m}[z]$ **function with power multipliers into the function of the same form.**

Also deduce Mittag-Leffler functions introduced by [1],[3],[5],[6],[7],[8],[9],[10],[14],[15],[16],[17] are particular cases of Multiparameter S-K-Mittag-Leffler function for some particular values of parameters and deduce some particular cases.

IndexTerms **- Multiparameter S-K-Mittag-Leffler function, K-Series, S-K-Pochhammer symbol, S-K- Gamma function, Riemann-Liouville fractional operators.**

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I. Introduction

In [4](Definition 2.2, Eqn.2.6) the author introduce the generalized S-K-Gamma Function ${}_s\Gamma_k(x)$ as

$$ {}_s\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty} \frac{n!s^{n+1}(ns)^{\frac{x}{k}-1}}{{}_s(x)_{n,k}},\ \ s,k>0, x \in C\backslash kZ^-, \tag{1} $$

where ${}_s(x)_{n,k}$ is the s-k-Pochhammer symbol [4](Definition 2.1, Eqn.2.1) and is given by

$$ {}_s(x)_{n,k} = (\tfrac{xs}{k})(\tfrac{xs}{k}+s)(\tfrac{xs}{k}+2s).....(\tfrac{xs}{k}+(n-1)s),\ \ x \in C, s,k \in R, n \in N^+. \tag{2} $$

Integral representation of S-K-Gamma function[4](Theorem 2.4, Eqn.2.14) is given by,

$$ {}_s\Gamma_k(x) = \int_0^{\infty} t^{x-1}e^{-\frac{t^k}{s}}dt, x \in C, k \in R, Re(x) > 0, \tag{3} $$

and it follows easily that [4](Eqns. 2.22 and 2.19)

$$ {}_s(x)_{n,k} = \frac{{}_s\Gamma_k(x+nk)}{{}_s\Gamma_k(x)}\ \ and\ \ {}_s\Gamma_k(x) = \left(\frac{s}{k}\right)^{\frac{x}{k}}\Gamma_k(x) = \frac{s^{\frac{x}{k}}}{k}\Gamma(\tfrac{x}{k}) \tag{4} $$

The Fractional Integral operators for $\alpha > 0$ ([11], Definition 2.1, Page 33) are defined as,

$$ (I_{0+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, (x>0), \tag{5} $$

$$ (I_{-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}}dt, (x>0), \tag{6} $$

The Fractional Derivative for $\alpha > 0$ ([11], Definition 2.2, Page 35) are defined as,

$$ (D_{0+}^{\alpha}f)(x) = (\tfrac{d}{dx})^{[\alpha]+1}(I_{0+}^{1-\{\alpha\}}f)(x) = \frac{1}{\Gamma(1-\{\alpha\})}(\tfrac{d}{dx})^{[\alpha]+1}\int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}}dt, (x>0), \tag{7} $$

$$ (D_{-}^{\alpha}f)(x) = (\tfrac{d}{dx})^{[\alpha]+1}(I_{-}^{1-\{\alpha\}}f)(x) = \frac{1}{\Gamma(1-\{\alpha\})}(-\tfrac{d}{dx})^{[\alpha]+1}\int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}}dt, (x>0). \tag{8} $$

where $[\alpha]$ means the maximal integer not extending α and $\{\alpha\}$ is the fractional part of α.

The next assertion is well known; see ([11], (2.44) and table 9.3, formula 1).
Let $\alpha \in C, (Re(\alpha) > 0)$ and $\gamma \in C$.

$$ if Re(\gamma) > 0, then (I_{0+}^{\alpha}t^{\gamma-1})(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}x^{\alpha+\gamma-1} \tag{9} $$

$$ if Re(\gamma) > Re(\alpha) > 0, then (I_{-}^{\alpha}t^{-\gamma})(x) = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}x^{\alpha-\gamma} \tag{10} $$

II. MAIN RESULT

In this section we introduce Multiparameter S-K-Mittag-Leffler function and for particular values of parameter we can deduce Multiparameter S-K-Mittag-Leffler function into eariler known Mittag-Leffler functions. Finally we calculate Fractional integral and derivative of Multiparameter S-K-Mittag-Leffler function.

2.1 Multiparameter S-K-Mittag-Leffer Function

Definition: Let $s, k \in R_+ = (0, \infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then the Multiparameter S-K-Mittag-Leffler function defnded as,

$$ {}_p^qK_{s,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)}, \qquad (11)$$

Where ${}_s\Gamma_k(x)$ is the S-K-Gamma function given by (1) and ${}_s(\gamma)_{n,k}$ is the S-K-Pochhammer symbol given by (2).

The series (11) is definded when none of the parameter $b_r (r = 1,2,..,q)$ is negative integer or zero. If any parameter $a_j (j = 1,2,..,p)$ in (11) is zero or negative, the series terminates into polynomial in z.

Convergent conditions for the series (11) are given by Ratio test,

(i) If $p < q + \sum_{i=1}^{m} (\frac{\eta_i}{k})$, then the power series on the right of (11) is absolutely convergent for all $z \in C$.

(ii) If $p = q + \sum_{i=1}^{m} (\frac{\eta_i}{k})$, then the power series on the right of (11) is absolutely convergent for all $|s^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})} z| < \prod_{i=1}^{m} (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$ and $|s^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})} z| = \prod_{i=1}^{m} (|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}, Re(\sum_{r=1}^{q} (\frac{b_r}{k}) + \sum_{i=1}^{m} (\frac{\beta_i}{k}) - \sum_{j=1}^{p} (\frac{a_j}{k})) > \frac{2+q+m-p}{2}$.

2.1.1 Particular cases

By particularizing the parameters in (11), we obtain following known Mittag-leffler functions.

(a) If we set $s = k$ in (11), we have

$$ {}_p^qK_{k,k}^{(\beta,\eta)_m}[z] = {}_p^qK_{k,k}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z],$$

$$ {}_pK_{q,k}^{(\beta,\eta)_m}[z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n,k}\, z^n}{\prod_{r=1}^{q} (b_r)_{n,k} \prod_{i=1}^{m} \Gamma_k(\eta_i n+\beta_i)},$$

Which is Multiparameter K-Mittag Leffler function definded by [3].

(b) If we set $s = k = 1$ in (11), we have

$$ {}_pK_{q,1}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n\, z^n}{\prod_{r=1}^{q} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_i n+\beta_i)},$$

$$ {}_pK_{q,1}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = {}_pK_q^{(\beta,\eta)_m}[z].$$

Which is K-Series defnded by [5],[6].

(c) If we set $s = k = 1, p = q = m$ and $b_1 = b_2 = \ldots = b_m = 1$ in (11), we have

$$ {}_mK_{m,1}^{(\beta,\eta)_m}[a_1,..,a_m; 1,1,..,1, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{m} (a_i)_n\, z^n}{\prod_{i=1}^{m} \Gamma(\eta_i n+\beta_i)(n!)^m},$$

$$ {}_mK_{m,1}^{(\beta,\eta)_m}[a_1,..,a_m; 1,1,..,1, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = E_{(\eta_i),(\beta_i)}^{(a_i),m}[z].$$

Which is the 3M-Parameter Multi-Index Mittag-Leffler function definded by [7].

(d) If we set $s = k = 1, p = q = 1, a_1 = \rho, b_1 = 1$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_m}[\rho; 1, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = \sum_{n=0}^{\infty} \frac{(\rho)_n\, z^n}{\prod_{i=1}^{m} \Gamma(\eta_i n+\beta_i)(n!)},$$

$$ {}_1K_{1,1}^{(\beta,\eta)_m}[\rho; 1, (\beta_1,\eta_1),..,(\beta_m,\eta_m); z] = E_\rho[(\eta,\beta)_m; z].$$

Which is the Generalized Mittag-Leffler function studied by [8].

(e) If we set $s = k = 1, p = q = 1, a_1 = b_1 = 1$ and $\eta_i = \frac{1}{\alpha_i}$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_m}[1; 1, (\beta_1,\frac{1}{\alpha_1}),..,(\beta_m,\frac{1}{\alpha_m}); z] = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{i=1}^{m} \Gamma(n\frac{1}{\alpha_i}+\beta_i)},$$

$$ {}_1K_{1,1}^{(\beta,\eta)_m}[1; 1, (\beta_1,\frac{1}{\alpha_1}),..,(\beta_m,\frac{1}{\alpha_m}); z] = E_{(\frac{1}{\alpha_i}),(\beta_i)}^{(m)}[z].$$

Which is the Multi-Index Mittag-Leffler function studied by [16].

(f) If we set $s = k = 1, m = 1$ in (11), we have

$$ {}_pK_{q,1}^{(\beta,\eta)_1}[a_1,..,a_p;b_1,..,b_q,(\beta,\eta);z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\, z^n}{\prod_{r=1}^{q}(b_r)_n \Gamma(\eta n+\beta)}. $$

$$ {}_pK_{q,1}^{(\beta,\eta)_1}[a_1,..,a_p;b_1,..,b_q,(\beta,\eta);z] = {}_pM_q^{\eta,\beta}[a_1,..,a_p;b_1,..,b_q;z]. $$

Which is Generalized M-Series definded by [15].

(g) If we set $s = k = 1, m = 1$ and $\beta = 1$ in (11), we have

$$ {}_pK_{q,1}^{(\beta,\eta)_1}[a_1,..,a_p;b_1,..,b_q,(1,\eta);z] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\, z^n}{\prod_{r=1}^{q}(b_r)_n \Gamma(\eta n+1)}. $$

$$ {}_pK_{q,1}^{(\beta,\eta)_1}[a_1,..,a_p;b_1,..,b_q,(1,\eta);z] = {}_pM_q^{\eta}[a_1,..,a_p;b_1,..,b_q;z]. $$

Which is M-Series definded by [14].

(h) If we set $s = k = 1, p = q = m = 1, a_1 = \delta, b_1 = k$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[\delta;k,(\beta,\eta);z] = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\eta n+\beta)(n!)} (\frac{z}{k})^n, $$

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[\delta;k,(\beta,\eta);z] = E_{k,\eta,\beta}^{\delta}[\frac{z}{k}]. $$

Which is the K- Mittag-Leffler function studied by [1].

(i) If we set $s = k = 1, p = q = m = 1, a_1 = \delta, b_1 = 1$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[\delta;1,(\beta,\eta);z] = \sum_{n=0}^{\infty} \frac{(\delta)_n\, z^n}{\Gamma(\eta n+\beta)(n!)}, $$

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[\delta;1,(\beta,\eta);z] = E_{\eta,\beta}^{\delta}[z]. $$

Which is the Generalized Mittag-Leffler function studied by [10].

(j) If we set $s = k = 1, p = q = m = 1, a_1 = b_1 = 1$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[1;1,(\beta,\eta);z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta n+\beta)}, $$

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[1;1,(\beta,\eta);z] = E_{\eta,\beta}[z]. $$

Which is the Mittag-Leffler function studied by [17].

(k) If we set $s = k = 1, p = q = m = 1, a_1 = b_1 = 1$ and $\beta = 1$ in (11),then we obtain

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[1;1,(1,\eta);z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\eta n+1)}, $$

$$ {}_1K_{1,1}^{(\beta,\eta)_1}[1;1,(1,\eta);z] = E_{\eta}[z]. $$

Which is the Mittag-Leffler function studied by [9].

2.2 Fractional Claculus of Multiparameter S-K-Mittag-Leffler Function

We will prove, the fractional integration and differentation of ${}_p^qK_{s,k}^{(\beta,\eta)_m}[z]$ function with power multipliers into the function of the same form.

Theorem 1. Let $\alpha > 0, a \in R, s, k \in R_+ = (0,\infty);\ a_j, b_r, \beta_i \in C;\ \eta_i \in R, Re(\beta_i) > 0;\ (j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then there holds the relation,

$$ (I_{0+}^{\alpha}[t^{\frac{\beta_1}{k}-1}\, {}_p^qK_{s,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x) = s^{\alpha} x^{(\frac{\beta_1}{k}+\alpha-1)}\, {}_p^qK_{s,k}^{(\beta,\eta)_m}[ax^{\frac{\eta_1}{k}}]. \quad (12) $$

Proof: By virtue of (5) and (11), we have

$$ A \equiv (I_{0+}^{\alpha}[t^{\frac{\beta_1}{k}-1}\, {}_p^qK_{s,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x), $$

$$ A \equiv (I_{0+}^{\alpha}[t^{\frac{\beta_1}{k}-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}\,(at^{\frac{\eta_1}{k}})^n}{\prod_{r=1}^{q}{}_s(b_r)_{n,k} \prod_{i=1}^{m}{}_s\Gamma_k(\eta_i n+\beta_i)}])(x), $$

on interchanging the order of the integration and summation, and using (9), we obtain

$$ A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}{}_s(a_j)_{n,k}}{\prod_{r=1}^{q}{}_s(b_r)_{n,k} \prod_{i=1}^{m}{}_s\Gamma_k(\eta_i n+\beta_i)} \frac{\Gamma(\frac{n\eta_1+\beta_1}{k}) a^n x^{(\frac{n\eta_1+\beta_1}{k}+\alpha-1)}}{\Gamma(\frac{n\eta_1+\beta_1}{k}+\alpha)}, $$

using (4) and rearranging the terms, we have

$$A \equiv s^{\alpha} x^{(\frac{\beta_1}{k}+\alpha-1)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=2}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} \frac{(ax^{\frac{\eta_1}{k}})^n}{{}_s\Gamma_k(n\eta_1+\beta_1+\alpha k)},$$

$$A \equiv s^{\alpha} x^{(\frac{\beta_1}{k}+\alpha-1)} {}_p^q K_{s,k}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1+\alpha k, \eta_1), (\beta_2,\eta_2)..,(\beta_m,\eta_m); ax^{\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 2. Let $\alpha > 0, a \in R, s,k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R, Re(\beta_i) > 0$; $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then there holds the relation,

$$(I_-^{\alpha}[t^{-\alpha-\frac{\beta_1}{k}} {}_p^q K_{s,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x) = s^{\alpha} x^{(-\frac{\beta_1}{k})} {}_p^q K_{s,k}^{(\beta,\eta)_m}[ax^{-\frac{\eta_1}{k}}]. \quad (13)$$

Proof. By virtue of (6) and (11), we have

$$A \equiv (I_-^{\alpha}[t^{-\alpha-\frac{\beta_1}{k}} {}_p^q K_{s,k}^{(\beta,\eta)_m}[at^{-\frac{\eta_1}{k}}]])(x),$$

$$A \equiv (I_-^{\alpha}[t^{-\alpha-\frac{\beta_1}{k}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} (at^{-\frac{\eta_1}{k}})^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)}])(x),$$

on interchanging the order of the integration and summation, and using (10), we obtain

$$A \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} \frac{\Gamma(\frac{n\eta_1+\beta_1}{k}) a^n x^{-(\frac{n\eta_1+\beta_1}{k})}}{\Gamma(\frac{n\eta_1+\beta_1}{k}+\alpha)},$$

using (4) and rearranging the terms, we have

$$A \equiv s^{\alpha} x^{(-\frac{\beta_1}{k})} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=2}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} \frac{(ax^{-\frac{\eta_1}{k}})^n}{{}_s\Gamma_k(n\eta_1+\beta_1+\alpha k)},$$

$$A \equiv s^{\alpha} x^{(-\frac{\beta_1}{k})} {}_p^q K_{s,k}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1+\alpha k, \eta_1), (\beta_2,\eta_2)..,(\beta_m,\eta_m); ax^{-\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 3. Let $\alpha > 0, a \in R, s,k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R, Re(\beta_i) > 0$; $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then there holds the relation,

$$(D_{0+}^{\alpha}[t^{\frac{\beta_1}{k}-1} {}_p^q K_{s,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x) = s^{-\alpha} x^{(\frac{\beta_1}{k}-\alpha-1)} {}_p^q K_{s,k}^{(\beta,\eta)_m}[ax^{\frac{\eta_1}{k}}]. \quad (14)$$

Proof. By virtue of (7) and (11), we have

$$A \equiv (D_{0+}^{\alpha}[t^{\frac{\beta_1}{k}-1} {}_p^q K_{s,k}^{(\beta,\eta)_m}[at^{\frac{\eta_1}{k}}]])(x),$$

$$A \equiv \frac{1}{\Gamma(\gamma-\alpha)} (\frac{d}{dx})^{\gamma} \int_0^x \frac{t^{\frac{\beta_1}{k}-1}}{(x-t)^{1+\alpha-\gamma}} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} (at^{\frac{\eta_1}{k}})^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} dt,$$

where $\gamma = [\alpha] + 1$

$$A \equiv \frac{1}{\Gamma(\gamma-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} a^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} (\frac{d}{dx})^{\gamma} \int_0^x \frac{t^{\frac{\beta_1+n\eta_1}{k}-1}}{(x-t)^{1+\alpha-\gamma}} dt,$$

put $t = ux$ the above expression transforms in to the form

$$A \equiv \frac{1}{\Gamma(\gamma-\alpha)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} a^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} (\frac{d}{dx})^{\gamma} x^{\frac{\beta_1+n\eta_1}{k}+\gamma-\alpha-1}$$

$$\times \int_0^1 (1-u)^{\gamma-\alpha-1} u^{\frac{\beta_1+n\eta_1}{k}-1} dt,$$

using the Beta function formula, we have

$$A \equiv x^{\frac{\beta_1}{k}-\alpha-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k} a^n}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=1}^{m} {}_s\Gamma_k(\eta_i n+\beta_i)} \frac{\Gamma(\frac{\beta_1+n\eta_1}{k})}{\Gamma(\frac{\beta_1+n\eta_1}{k}-\alpha)} x^{\frac{n\eta_1}{k}},$$

using (4) and rearranging the terms, we obtain

$$A \equiv s^{-\alpha} x^{\frac{\beta_1}{k}-\alpha-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} {}_s(a_j)_{n,k}}{\prod_{r=1}^{q} {}_s(b_r)_{n,k} \prod_{i=2}^{m} {}_s\Gamma_k(\eta_i n+\beta_i) {}_s\Gamma_k(n\eta_1+\beta_1-k\alpha)} (ax^{\frac{\eta_1}{k}})^n,$$

$$A \equiv s^{-\alpha} x^{\frac{\beta_1}{k}-\alpha-1} {}_p^q K_{s,k}^{(\beta,\eta)_m}[a_1,..,a_p; b_1,..,b_q, (\beta_1-\alpha k, \eta_1), (\beta_2,\eta_2)..,(\beta_m,\eta_m); ax^{\frac{\eta_1}{k}}],$$

which proves the theorem.

Theorem 4. Let $\alpha > 0, a \in R, s,k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R, Re(\beta_i) > 0$; $(j = 1,2,..,p; r = 1,2,..,q; i = 1,2,..,m)$. Then there holds the relation,

$$(D_{-}^{\alpha}[t^{\alpha-\frac{\beta_1}{k}}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)m}[at^{-\frac{\eta_1}{k}}]])(x) = s^{-\alpha}x^{(-\frac{\beta_1}{k})}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)m}[ax^{-\frac{\eta_1}{k}}]. \quad (15)$$

Proof. By virtue of (8) and (11), we have

$$A \equiv (D_{-}^{\alpha}[t^{\alpha-\frac{\beta_1}{k}}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)m}[at^{-\frac{\eta_1}{k}}]])(x),$$

$$A \equiv \frac{1}{\Gamma(\gamma-\alpha)}(-\frac{d}{dx})^{\gamma}\int_x^{\infty}\frac{t^{\alpha-\frac{\beta_1}{k}}}{(t-x)^{1+\alpha-\gamma}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,(at^{-\frac{\eta_1}{k}})^n}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}{}_{s}\Gamma_k(\eta_i n+\beta_i)}dt,$$

where $\gamma = [\alpha] + 1$

$$A \equiv \frac{1}{\Gamma(\gamma-\alpha)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,a^n}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}{}_{s}\Gamma_k(\eta_i n+\beta_i)}(-\frac{d}{dx})^{\gamma}\int_x^{\infty}\frac{t^{\alpha-\frac{\beta_1+n\eta_1}{k}}}{(t-x)^{1+\alpha-\gamma}}dt,$$

put $t = \frac{x}{u}$ and using the Beta function, the above expression transforms in to the form,

$$A \equiv x^{-\frac{\beta_1}{k}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,a^n\Gamma(\frac{\beta_1+n\eta_1}{k}-\gamma)(-1)^{\gamma}\Gamma(1+\gamma-\frac{\beta_1+n\eta_1}{k})x^{-\frac{n\eta_1}{k}}}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}{}_{s}\Gamma_k(\eta_i n+\beta_i)\Gamma(\frac{\beta_1+n\eta_1}{k}-\alpha)\Gamma(1-\frac{\beta_1+n\eta_1}{k})}, \quad (16)$$

the reflection formula for gamma function, see ([11],1.60),

$$\frac{1}{\Gamma(1-\frac{\beta_1+n\eta_1}{k})} = \frac{\Gamma(\frac{\beta_1+n\eta_1}{k})}{\Gamma(\frac{\beta_1+n\eta_1}{k})\Gamma(1-\frac{\beta_1+n\eta_1}{k})} = \frac{\Gamma(\frac{\beta_1+n\eta_1}{k})\sin[(\frac{\beta_1+n\eta_1}{k})\pi]}{\pi}, \quad (17)$$

and

$$\Gamma(\frac{\beta_1+n\eta_1}{k}-\gamma)\Gamma(1+\gamma-\frac{\beta_1+n\eta_1}{k}) = \frac{\pi}{\sin[(\frac{\beta_1+n\eta_1}{k}-\gamma)\pi]},$$

$$= \frac{\pi}{\sin[(\frac{\beta_1+n\eta_1}{k})\pi]\cos(\gamma\pi)} = \frac{\pi(-1)^{\gamma}}{\sin[(\frac{\beta_1+n\eta_1}{k})\pi]}, \quad (18)$$

using (17) and (18) in (16), we obtain

$$A \equiv x^{-\frac{\beta_1}{k}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}\,a^n\Gamma(\frac{\beta_1+n\eta_1}{k})x^{-\frac{n\eta_1}{k}}}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=1}^{m}{}_{s}\Gamma_k(\eta_i n+\beta_i)\Gamma(\frac{\beta_1+n\eta_1}{k}-\alpha)},$$

using (4) and rearranging the terms, we obtain

$$A \equiv s^{-\alpha}x^{-\frac{\beta_1}{k}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}{}_{s}(a_j)_{n,k}}{\prod_{r=1}^{q}{}_{s}(b_r)_{n,k}\prod_{i=2}^{m}{}_{s}\Gamma_k(\eta_i n+\beta_i){}_{s}\Gamma_k(n\eta_1+\beta_1-k\alpha)}(ax^{-\frac{\eta_1}{k}})^n,$$

$$A \equiv s^{-\alpha}x^{-\frac{\beta_1}{k}}\,{}_{p}^{q}K_{s,k}^{(\beta,\eta)m}[a_1,..,a_p;b_1,..,b_q,(\beta_1-\alpha k,\eta_1),(\beta_2,\eta_2)..,(\beta_m,\eta_m);ax^{-\frac{\eta_1}{k}}],$$

which proves the theorem.

2.2.1 Particular Cases

(A) By putting $s = k$,
(i) in equation (12), then it reduces to the result earlier given by ([3], Theorem 1, Equation 12).
(ii) in equation (13), then it reduces to the result earlier given by ([3],Theorem 2, Equation 13).
(iii) in equation (14), then it reduces to the result earlier given by ([3],Theorem 3, Equation 14).
(iv) in equation (15), then it reduces to the result earlier given by ([3],Theorem 4, Equation 15).

(B) By putting $s = k = 1$,
(i) in equation (12), then it reduces to the result earlier given by ([5], Theorem 3.1, Equation 6).
(ii) in equation (13), then it reduces to the result earlier given by ([5],Theorem 3.2, Equation 7).
(iii) in equation (14), then it reduces to the result earlier given by ([5],Theorem 3.3, Equation 8).
(iv) in equation (15), then it reduces to the result earlier given by ([5],Theorem 3.4, Equation 9).

(C) By putting $s = k = 1, p = q = m$ and $b_1 = b_2 = ... = b_m = 1$,
(i) in equation (12), then it reduces to the result earlier given by ([7], Theorem 4.1, Equation 4.4).
(ii) in equation (14), then it reduces to the result earlier given by ([7],Theorem 4.2, Equation 4.5).

(D) By putting $s = k = p = q = 1, a_1 = \rho$ and $b_1 = 1$,
(i) in equation (12), then it reduces to the result earlier given by ([12]], Theorem 1, Equation 2.1).
(ii) in equation (13), then it reduces to the result earlier given by ([12],Theorem 3, Equation 2.4).
(iii) in equation (14), then it reduces to the result earlier given by ([12],Theorem 5, Equation 2.6).
(iv) in equation (15), then it reduces to the result earlier given by ([12],Theorem 7, Equation 2.8).

(E) By putting $s = k = p = q = m = 1, a_1 = \delta$ and $b_1 = 1$,
(i) in (12), then it reduces to the result earlier given by ([13], Theorem 1, Equation 14).
(ii)in (13) then it reduces to the result earlier given by ([13],Theorem 3, Equation 23).
(iii)in (14) then it reduces to the result earlier given by ([13],Theorem 5, Equation 29).

(iv)in (15) then it reduces to the result earlier given by ([13],Theorem 7, Equation 35).

(F) By putting $s = k = m = 1$ and $\beta = 1$,
(i) in (12), then it reduces to the result earlier given by ([2], Theorem 2.1, Equation 6).
(ii)in (13) then it reduces to the result earlier given by ([2],Theorem 2.2, Equation 10).
(iii)in (14) then it reduces to the result earlier given by ([2],Theorem 2.3, Equation 14).
(vi)in (15) then it reduces to the result earlier given by ([2],Theorem 2.4, Equation 18).

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