Mohand Decomposition Method for Solving Nonlinear Pantograph Delay Differential Equations

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Abstract: This paper presents Mohand Decomposition Method for the solution of nonlinear Pantograph Delay Differential Equations. This method is a combination of Mohand transform and Adomian decomposition method. For this, the solution is obtained as a series by first applying the Mohand Transform to the DDEs and then decomposing the nonlinear term by finding Adomian polynomials. Numerical examples are given to illustrate the effectiveness of our proposed method.

***Key words* - Mohand Integral Transform, Adomian Polynomial, Nonlinear Pantograph Delay Differential Equations**

I. INTRODUCTION

Delay Differential Equations (DDEs) are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. DDEs appear in chemical kinetics [1], population dynamics [2], traffic models [3] and control systems [4] and in several fields. The general theories of DDEs have been widely developed by Bellman and Cooke [5], Hale [6], Driver [7] and Norkin [8].

The pantograph equation is special type of delay differential equations with proportional delays. Pantograph is a device located on the electric locomotive. The name pantograph originated from the study by Ockendon and Tayler [9].
These equations arise in different fields such as number theory, probability, electrodynamics, astrophysics, and quantum mechanics. The first order pantograph equation is given as

$$y'(t) = ay(t) + by(qt), \quad y(t) = \emptyset(t) \tag{1}$$

Here, $\emptyset(t)$ is the initial function and $0 < q < 1$.

Several analytical and numerical methods have been proposed by the researchers for the solution of DDEs of pantograph type. Some of them are variational iteration method [10], perturbation-iteration method [11], differential transformation method [12], Runge-Kutta method [13] and Chebyshev series [14].

In this paper, Mohand Decomposition method is proposed to solve the nonlinear pantograph DDEs. This method is the combination of Mohand transform and Adomian decomposition method which is capable to solve nonlinear pantograph delay differential equations. This paper has been organized as follows: In Section 2, the Mohand transform and its fundamental properties have been discussed. In Section 3, the analysis of the proposed method has been given for nonlinear pantograph DDEs. In Section 4, numerical examples have been provided to demonstrate the efficiency of the proposed method.

II. MOHAND TRANSFORM AND ITS FUNDAMENTAL PROPERTIES

Mohand transform was recently introduced by Mohand Mahgoub [15] in 2017. This transform is derived from the classical Fourier integral. It is defined for function of exponential order in the set $\boldsymbol{A}$ defined by:

$$A = \{f(t): \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}\}$$

For a given function in the set A, M is constant and finite number, k_1, k_2 either finite or infinite. The Mohand Transform is denoted by the operator $M(\cdot)$ defined by the integral equation:

$$M[f(t)] = R(v) = v^2 \int_0^\infty f(t)e^{-vt}dt, \ t \geq 0, \ k_1 \leq v \leq k_2 \tag{2}$$

In this transform the variable v is used to factor the variable t in the argument of the function f. This transform has connection with Fourier and Laplace transforms.

Mohand Transform of simple functions is given below:

i. $M[1] = v$
ii. $M[t] = 1$
iii. $M[t^n] = \frac{n!}{v^{n-1}}$ where n is the positive integer.
iv. $M[e^{at}] = \frac{v^2}{v-a}$
v. $M[\sin at] = \frac{av^2}{v^2+a^2}$
vi. $M[\cos at] = \frac{v^3}{v^2+a^2}$

Mohand Transform for derivatives are:

i. $M[f'(t)] = vR(v) - v^2 f(0)$
ii. $M[f''(t)] = v^2 R(v) - v^3 f(0) - v^2 f'(0)$
iii. $M\left[f^{(n)}(t)\right] = v^{(n)} R(v) - \sum_{k=0}^{n-1} v^{n-k+1} f^{(k)}(0)$

III. ANALYSIS OF THE PROPOSED METHOD FOR NONLINEAR PANTOGRAPH DDES

The Adomian Decomposition Method (ADM) has been developed by Adomian [16]. Mohand Decomposition Method, which is the combination of Mohand Transform and ADM, has been proposed here to solve the nonlinear DDEs. The solution is obtained as a series. First, we apply the Mohand transform to DDEs, then decomposing the nonlinear term by finding Adomian polynomials.

Consider the first order nonlinear pantograph DDE of the form

$$y' = f(t, y, y(qt)), \quad y(0) = \alpha \tag{3}$$

Apply the Mohand transform on both sides of (3) and using initial condition we get,

$$M[y'] = M[f(t, y, y(qt))]$$

$$M[y(t)] = \alpha v + \frac{1}{v} M\left[f\left(t, y, y(qt)\right)\right]$$

The nonlinear term $N(y, y_\tau)$ can be decomposed into an infinite series of Adomian polynomials as $\sum_{n=0}^{\infty} A_n$, where A_n is classically suggested to the computed form

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i y_i, \sum_{i=0}^{\infty} \lambda^i y_{\tau i})]_{\lambda>0}$$

The proposed technique now gives the solution as an infinite series,

$$M[\sum_{n=0}^{\infty} y_n] = \alpha v + \frac{1}{v} M[\sum_{n=0}^{\infty} A_n] \tag{4}$$

where A_n is Adomian polynomials for nonlinear terms.
From Eqn. (4), we get the following recursive algorithm

$$M[y_0] = \alpha v$$

$$M[y_{n+1}] = \frac{1}{v} M[A_n], \; n > 0.$$

By taking inverse transform, we get $y_0, y_1, y_2, \dots ....$

The analytical solution of nonlinear pantograph DDEs by is given as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t).$$

By finding sufficient number of $y_n{}'s$ we get the numerical solution with good accuracy.

IV. NUMERICAL EXAMPLES

EXAMPLE 4.1:
Consider the first order nonlinear pantograph DDE

$$y'(t) = 2 - y^2\left(\frac{t}{2}\right), \quad y(0) = 0$$

The exact solution is $y(t) = 2\sin(t)$.
Applying Mohand Transform to the above first order pantograph nonlinear DDE, we obtain

$$M[y(t)] = 2 - \frac{1}{v} M\left[y^2\left(\frac{t}{2}\right)\right]$$

Then using Adomian Decomposition Method, we obtain

$$M[\sum_{n=0}^{\infty} y_n(t)] = 2 - \frac{1}{v} M[\sum_{n=0}^{\infty} A_n] \tag{5}$$

From Eqn. (5), we get,

$$y_0(t) = 2t$$

$$y_1(t) = -\frac{2t^3}{3!},$$

$$y_2(t) = \frac{2t^5}{5!},$$

$$y_3(t) = -\frac{2t^7}{7!},$$

… … ….

The infinite series solution becomes,

$$y = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$y = 2t - \frac{2t^3}{3!} + \frac{2t^5}{5!} - \frac{2t^7}{7!} + \cdots$$

which converges to the exact solution $y(t) = 2\sin(t)$ as $n \to \infty$.

EXAMPLE 4.2:

Consider the second order pantograph nonlinear DDE

$$y''(t) = -y(t) + 5y^2\left(\frac{t}{2}\right),$$

$$y(0) = 1, \qquad y'(0) = -2$$

The exact solution is $y(t) = e^{-2t}$.

Applying Mohand Transform to the above second order pantograph nonlinear DDE, we obtain

$$M[y(t)] = -2 + v - \frac{1}{v^2}M[y(t)] + \frac{5}{v^2}M\left[y^2\left(\frac{t}{2}\right)\right]$$

Then using Adomian Decomposition Method, we obtain

$$M[\sum_{n=0}^{\infty} y_n(t)] = -2 + v - \frac{1}{v^2}M[\sum_{n=0}^{\infty} A_n] + \frac{5}{v^2}M[\sum_{n=0}^{\infty} A_n] \quad (6)$$

From Eqn. (6), we get

$$y_0(t) = 1 - 2t$$

$$y_1(t) = 2t^2 - \frac{4t^3}{3} + \frac{5t^4}{12},$$

$$y_2(t) = \frac{t^4}{4} - \frac{t^5}{15} + \frac{9t^6}{320} - \frac{55t^7}{4032},$$

… … ….

The infinite series solution becomes,

$$y = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$y = 1 - 2t + 2t^2 - \frac{4t^3}{3} + \frac{2t^4}{3} - \frac{4t^5}{15} \ldots$$

which converges to the exact solution $y(t) = e^{-2t}$ as $n \to \infty$.

CONCLUSION

In this paper we proposed Mohand Decomposition Method to solve nonlinear pantograph delay differential equations. It is the combination of Mohand transform and Adomian decomposition method to produce exact/approximate solutions of the nonlinear DDEs. This method gives rapidly convergent successive approximations by recursive relations. Two examples have been considered to demonstrate the applicability of the proposed method. This method is very efficient to solve nonlinear pantograph DDEs and gives results with good accuracy.

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