Thermal Stresses of a Solid Cylinder with Internal Heat Source: Direct Problem

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Abstract: --The aim of this paper is to study thermal stresses of a circular cylinder, in which boundary conditions are of radiation type. We apply integral transform techniques and obtained the solution of the problem. Numerical calculations are carried out for a particular case and results are depicted graphically.

Keywords: **--Thermoelastic response, Circular cylinder, integral transform, thermal stress.**

Introduction

Deshmukh et al. [1] have studied quasi – static thermal stresses in a thick circular plate. Kamdi et al. [2] have discussed Transient thermoelastic problem for a circular solid cylinder with radiation. Nowacki [3] has studied the state of stress in a thick circular plate due to temperature field. Roy Choudhary [6] has derived quasi – static thermal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. Wankhede [7] has discussed the quasi – static thermal stresses in a circular plate. Khobragade et al. [8] have studied Transient thermoelastic problem of semi-Infinite circular beam with internal heat sources. Meshram et al. [9] have discussed steady state thermoelastic problems of semi-Infinite hollow cylinder on outer curved surface.This paper is concerned with the transient thermoelastic problem of a solid cylinder occupying the space $0 \le r \le a$, $-h \le z \le h$ with radiation type boundary conditions.

Statement of The Problem

Consider a circular solid cylinder occupying the space $D = \{x, y, z) \in R^3 ; 0 \le (x^2 + y^2)^{1/2} \le a ; -h \le z \le h\}$ where $r = (x^2 + y^2)^{1/2}$. The material is isotropic homogeneous and all properties are assumed to be constant. Heat conduction with internal heat source and the prescribed boundary conditions of the radiation type are considered. The equation for heat conduction is [4]

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]+\chi(r,z,t)=\frac{\partial T}{\partial t} \tag{2.1}$$

Where k is the thermal diffusivity of the material of the cylinder (which is assumed to be constant).

Subject to the initial and boundary condition

$$M_t(T,1,0,0)=0 \text{, for all } 0 \le r \le a, -h \le z \le h \tag{2.2}$$

$$M_r(T,1,0,a)=0 \text{, for all } -h \le z \le h, \ t > 0 \tag{2.3}$$

$$M_z(T,1,k_1,h)=\left(-\frac{Q_0}{\lambda}\right) f(r,t), \text{ for all } 0 \le r \le a, \ t > 0 \tag{2.4}$$

$$M_z(T,1,k_{2,}-h)=g(r,t) \text{, for all } 0 \le r \le a, \ t > 0 \tag{2.5}$$

The most general expression for these conditions can be given by

$$M_v(f,\bar{k},\bar{\bar{k}},\$) = (\bar{k}f + \bar{\bar{k}}\hat{f})_{v=\$}$$

where the prime $(\wedge)$ denotes differentiation with respect to v; $\bar{k}$ and $\bar{\bar{k}}$ are radiation constants on the upper and lower surface of cylinder respectively.

The Navier's equations without the body forces for axisymmetric two dimensional thermoelastic problem can be expressed as [3]

$$\nabla^2 u_r - \frac{u_r}{r^2} + \frac{1}{1-2v}\frac{\partial e}{\partial r} - \frac{2(1+v)}{1-2v}\alpha_t \frac{\partial T}{\partial r} = 0 \tag{2.6}$$

$$\nabla^2 u_z - \frac{1}{1-2v}\frac{\partial e}{\partial z} - \frac{2(1+v)}{1-2v}\alpha_t \frac{\partial T}{\partial z} = 0 \tag{2.7}$$

where u_r and u_z are the displacement components in the radial and axial directions respectively and the dilation e as

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_r}{\partial z} \tag{2.8}$$

The displacement functions in the cylindrical coordinate system are represented by the Goodier'sthermoelastic displacement potential $\phi(r,z,t)$ and Love's function L as [4]

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \tag{2.9}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-v)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \tag{2.10}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \left(\frac{1+v}{1-v}\right)\alpha_t T \tag{2.11}$$

and the Love's function L must satisfy the equation

$$\nabla^2(\nabla^2 L) = 0 \tag{2.12}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$$

The component of the stresses are represented by the use of the potential ϕ and Love's function L as [4]

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(v\,\nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\} \tag{2.13}$$

$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(v\,\nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right)\right\} \tag{2.14}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left((2-v)\,\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \tag{2.15}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r\,\partial z} + \frac{\partial}{\partial r}\left((1-v)\,\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \tag{2.16}$$

Where G and v are the shear modulus and Poisson's ratio respectively.

The boundary conditions on the traction free surface of a solid cylinder are

$$\sigma_{rr} = \sigma_{rz} = 0 \ \ at \ \ r = a \tag{2.17}$$

Equations (2.1) to (2.17) constitute the mathematical formulation of the problem under consideration.

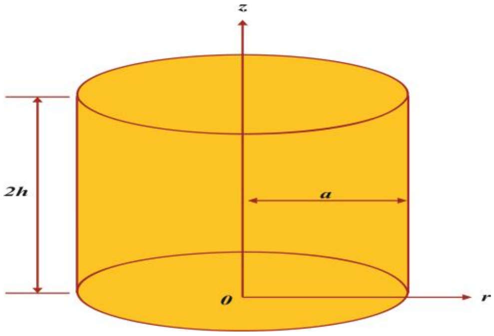


Figure shows the geometry of the problem

Solution of the Problem

Applying finite Hankel transform as [5] to the equations (2.4), (2.5) and (2.7) and using equation (2.6), one obtains

$$k\left[-\xi_n^2 T^*(\xi_n, z, t) + \frac{\partial^2 T^*(\xi_n, z, t)}{\partial z^2}\right] + \chi^* = \frac{\partial T^*(\xi_n, z, t)}{\partial t} \tag{3.1}$$

$$M_t(T^*, 1, 0, 0) = 0 \tag{3.2}$$

$$M_z(T^*, 1, k_1, h) = \left(-\frac{Q_0}{\lambda}\right) f^* \tag{3.3}$$

$$M_z(T^*, 1, k_2, -h) = g^* \tag{3.4}$$

where the symbol (*) means a function in the transformed domain and the nucleus for the finite Hankel transform defined by

$$K_0(\xi_n \, r) = \frac{-\sqrt{2}}{a}\left(\frac{J_0(\xi_n r)}{\xi_n \; J_0(\xi_n a)}\right) \tag{3.5}$$

The eigen values ξ_n are the positive roots of the characteristic equation

$$J_0(\xi_n a) = 0 \tag{3.6}$$

and $J_n(x)$ is the Bessel's function of the first kind of order n.

Further applying finite Marchi-Fasulo transform to the equation (3.1) and using equations (3.3), (3.4), one obtains

$$k\left[-\xi_n^2 \overline{T}^*(\xi_n, m, t) + \left[\frac{P_m(h)}{k_1}\left(\frac{Q_0}{\lambda}\right) f^* - \frac{P_m(-h)}{k_2} g^*\right] - \mu_m^2 \overline{T}^*(\xi_n, m, t)\right] + \overline{\chi}^* = \frac{d\overline{T}^*(\xi_n, m, t)}{dt} \tag{3.7}$$

$$M_t(\overline{T}^*, 1, 0, 0) = 0 \tag{3.8}$$

where $\overline{T}^*$ is the transformed function of $\overline{T}$ and m is the transformed parameter. The symbol (–) means a function of the transformed domain and the nucleus is given by the orthogonal functions in the interval $-h \le z \le h$ as

$$P_m(z) = Q_m \ \cos(\mu_m z) - W_m \ \sin(\mu_m z)$$

In which

$$Q_m = \mu_m (k_1 + k_2) \cos(\mu_m h)$$

$$W_m = 2\cos(\mu_m h) + (k_2 - k_1)\, \mu_m \sin(\mu_m h)$$

$$\lambda_m^2 = \int_{-h}^{h} P_m^2(z)\, dz = h[Q_m^2 + W_m^2] + \frac{\sin(2\mu_m h)}{2\mu_m}[Q_m{}^2 - W_m^2]$$

The eigen values μ_m are the positive roots of the characteristic equation

$$[k_1 a \cos(ah) + \sin(ah)][\cos(ah) + k_2 a \sin(ah)]$$

$$= [k_2 a \cos(ah) - \sin(ah)][\cos(ah) - k_1 a \sin(ah)]$$

After performing calculations on the equation (3.7), the reduction is made to linear first order differential equation as

$$\frac{d\overline{T}^*(\xi_n, m, t)}{dt} + k\, \Lambda_{m,n}\, \overline{T}^* = \Omega(\xi_n, \mu_m) \quad (3.9)$$

where

$$\Lambda_{m,n} = \mu_m^2 + \xi_n^2 \quad (3.10)$$

and

$$\Omega(\xi_n, \mu_m) = k\left[\left[\frac{P_m(h)}{k_1}\left(\frac{Q_0}{\lambda}\right) f^* - \frac{P_m(-h)}{k_2} g^*\right] + \overline{\chi}^*\right] \quad (3.11)$$

The transformed temperature solution of differential equation (3.9) is

$$\overline{T}^* = \frac{\Omega(\xi_n, \mu_m)}{(k\ \Lambda_{m,n} t)}[1 - e^{-(k\,\Lambda_{m,n} t)}] \quad (3.12)$$

Applying the inversion theorems of transformation rules to the equation (3.12), one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Psi_{n,m}[1 - \exp(-k\,\Lambda_{m,n} t)] \times P_m(z)\, K_0(\xi_n r) \quad (3.13)$$

where

$$\Psi_{n,m} = \frac{\Omega(m,n)}{(k\,\Lambda_{m,n} t)\, \lambda_m} \quad (3.14)$$

Equation (3.13) represents the temperature at any instant and at all points of a circular cylinder when there are radiation type boundary conditions.

15. Thermoelastic Displacement

Referring to the fundamental equation (2.1) and its solution (3.13) for the heat conduction problem, the solution for the displacement functions are represented by Goodier'sthermoelastic potential ϕ governed by the equation (2.11) as

$$\phi(r,z,t)=-\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]\times P_m(z)\,K_0(\xi_n r) \quad (4.1)$$

Similarly the solutions for Love's function L are assumed so as to satisfy the governed condition of equation (2.12) as

$$L(r,z,t)=-\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times[A_n J_0(\xi_n r)+C_n(\xi_n r)\,J_1(\xi_n r)]\times[\cos h(\xi_n z)] \quad (4.2)$$

Using (4.1) and (4.2) in (2.9) and (2.10), one obtains

$$u_r=\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times\left[\,[\xi_n\sin h(\xi_n z)][A_n(-\xi_n)J_1(\xi_n r)+C_n(\xi_n r)J_0(\xi_n r)]-\frac{\sqrt{2}}{a}\frac{J_1(\xi_n r)}{J_0(\xi_n a)}P_m(z)\right] \quad (4.3)$$

$$u_z=\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$[\mu_m[Q_m\sin(\mu_m z)+W_m\cos(\mu_m z)]\,K_0(\xi_n,r)+[A_n\xi_n^2 J_0(\xi_n r)][\cos h\,(\xi_n z)]$$

$$-C_n\xi_n^2[4(1-v)J_0(\xi_n r)-(\xi_n r)\,J_1(\xi_n r)]\times[\cos h(\xi_n z)] \quad (4.4)$$

Then the stress components can be evaluated by substituting the values of thermoelastic displacement potential ϕ from equation (4.1) and Love's function L from equation (4.2) in equations (2.13) to (2.16), one obtain

$$\sigma_{rr}=-2G\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times[(\mu_m^2+2\xi_n^2)\,J_0(\xi_n r)-\xi_n^2 J_1(\xi_n r)]\left[\frac{\sqrt{2}}{a\,\xi_n J_0(\xi_n a)}\right]P_m(z)$$

$$-A_n\xi_n^2\big[\xi_n J_1(\xi_n r)-J_0(\xi_n r)\big][\xi_n\sin h(\xi_n z)]+C_n\xi_n^2[(2v-1)J_0(\xi_n r)$$

$$+(\xi_n r)J_1(\xi_n r)][\xi_n\sin h\,(\xi_n z)] \quad (4.5)$$

$$\sigma_{\theta\theta}=-2G\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times\big[\Lambda_{m,n}J_0(\xi_n r)+\xi_n J_1(\xi_n r)\big]\left[\frac{\sqrt{2}}{a\,\xi_n J_0(\xi_n a)}\right]P_m(z)$$

$$+\big[A_n\xi_n^2 J_1(\xi_n r)[\sin h(\xi_n z)\big]+C_n\xi_n^3[(2v-1)J_0(\xi_n r)][\sin h\,(\xi_n z)] \quad (4.6)$$

$$\sigma_{zz}=-2G\left(\frac{1+v}{1-v}\right)\alpha_t\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times\ [-2\mu_m^2 - \xi_n^2]\, P_m(z)\, K_0(\xi_n, r) - [A_n^2 \xi_n^3 J_0(\xi_n r)\, [\sin h\, (\xi_n z)]$$

$$-C_n\ \xi_n^3 [2(v-2)\ J_0(\xi_n r) - (\xi_n r) J_1(\xi_n r)][\sin h\, (\xi_n z)] \tag{4.7}$$

$$\sigma_{rz} = -2G\left(\frac{1+v}{1-v}\right)\alpha_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Psi_{n,m}}{\Lambda_{m,n}}\times[1-\exp(-k\,\Lambda_{m,n}t)]$$

$$\times[-\mu_m^2][Q_m \sin(\mu_m z) + W_m \cos(\mu_m z)] \times \left[\frac{\sqrt{2}\ J_1(\xi_n r)}{a\ J_0(\xi_n a)}\right] + [A_n \xi_n^3 J_1(\xi_n r)][\cos h\, (\xi_n z)]$$

$$-C_n[\xi_n^3 J_0(\xi_n r) + 2(1-v)\ \xi_n^2 J_1(\xi_n r)][\cos h\, (\xi_n z)] \tag{4.8}$$

Determination of Unknown Arbitrary Functions an AndCn

Applying boundary conditions (2.17) to the equations (4.1) and (4.2), one obtains

$$A_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left[\sqrt{2}\ \hat{D}\ a^{-1}\xi_n (B_0 + C_0) \sin h\,(\xi_n z) - D_0 E_0 (\hat{B}+\hat{C}) \cos h\,(\xi_n z)\, P_m(z)\right]}{\left[\xi_n^3[\xi_n (B_0 + C_0)\hat{A} - (\hat{B}+\hat{C})A_0] \times \cos h(\xi_n z) \sin h\,(\xi_n z)\right]}$$

$$C_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left[D_0 E_0 \hat{A} P_m(z) \cos h(\xi_n z) - \sqrt{2} a^{-1}\hat{D} A_0\ \sin h(\xi_n z)\right]}{\left[\xi_n^2[\xi_n (B_0 + C_0)\hat{A} - (\hat{B}+\hat{C})A_0] \times \cos h\,(\xi_n z) \sin h(\xi_n z)\right]}$$

where

$$A_0 = \left[\frac{J_1(\xi_n a)}{\xi_n a} - J_0(\xi_n a)\right]$$

$$B_0 = (2v-1) J_0(\xi_n a)$$

$$C_0 = (\xi_n a)\ J_0(\xi_n a)$$

$$D_0 = \mu_m^2 + 2\xi_n^2 - \xi_n$$

$$E_0 = \left[\frac{J_0(\xi_n a)}{\xi_n J_0(\xi_n a)} - \xi_n\right]\frac{\sqrt{2}}{a}$$

$$\hat{A} = J_1(\xi_n a)\ \hat{B} = \xi_n J_0(\xi_n a)\ \hat{C} = 2(1-v)\ J_0(\xi_n a)$$

$$\hat{D} = \mu_m^2 [Q_m \sin(\mu_m z) + W_m \cos(\mu_m z)]\frac{\sqrt{2}}{a}$$

Special Case And Numerical Results

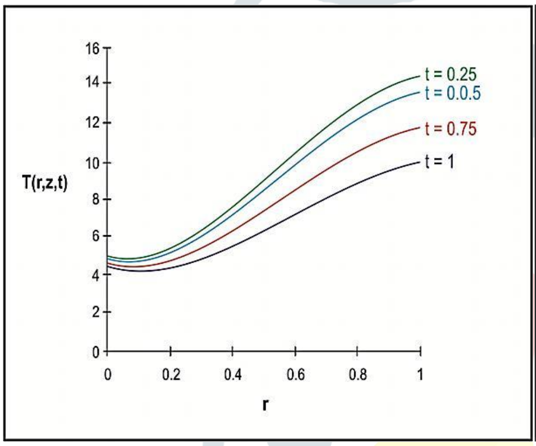
Set $f(r,t) = r(1-e^{-t})e^h$, $g(r,t) = r(1-e^{-t})e^{-h}$, $\chi(r,z,t) = \delta(r-r_0)\delta(z-z_0)\delta(t-t_0)$ (6.1)

Modules of elasticity, E (dynes/cm2)	6.9 × 1011
Shear modulus, G (dynes/cm2)	2.7 × 1011
Poisson ratio v	0.281

Thermal expansion coefficient, α_t (cm/cm-0C)	25.5 × 10–6
Thermal diffusivity, k (cm2/sec)	0.86
Thermal conductivity λ, (Cal – cm/ 0C/sec/cm2	0.48
Outer radius, a (cm)	5
Height, h (cm)	100

Conclusion

The temperature distributions, displacement and stress functions at any point of a solid cylinder have been determined where the cylinder is subjected to known heat source function . The integral transform methods have been used to obtain the solution of the problem. The results are obtained in terms of Bessel's function in the form of infinite series. The expressions are represented graphically.



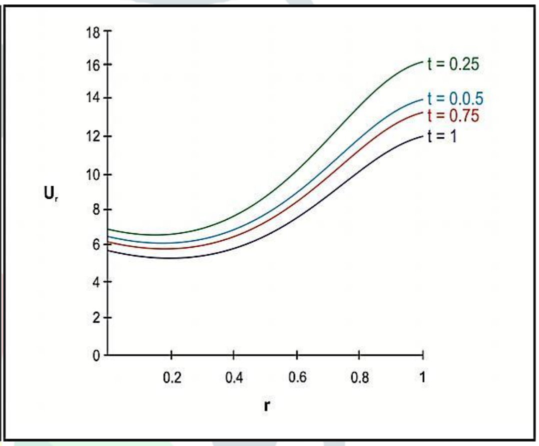
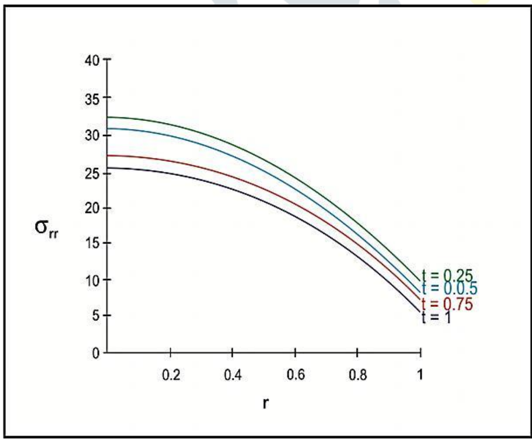


Figure (1) : Graph of r vs T(r,z,t) **Figure (2) : Graph of r vsur**



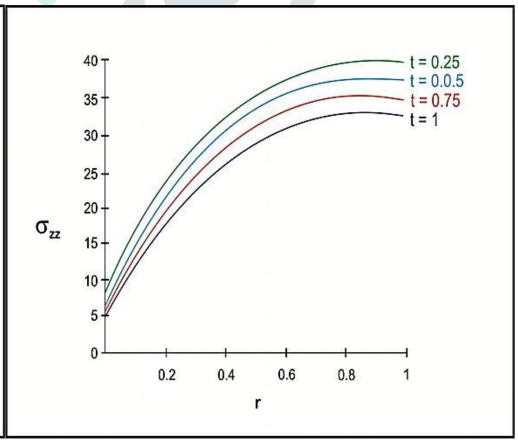


Figure (3) : Graph of r vsσ_{rr} Figure (4) : Graph of r vsσ_{zz}

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