Generalized Pre-Closed Sets via Ideals

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Abstract: -- The purpose of this paper is to study a class of closed sets, called generalized pre-closed sets with respect to an ideal (briefly Igp-closed sets), which is an extension of generalized pre-closed sets in general topology. Then, by using these sets, some classes of maps like continuous and closed maps via ideals have been introduced and analogues of some known results for continuous maps and closed maps in general topology have been obtained.

Keywords: **--Ideal, gp-closed sets, gp-closed maps, gp-continuous maps.**

Introduction

The closed sets are very important in topology. In 1970, Levine [9] introduced the concept of generalized closed sets. A.S. Mashhour [10] et. al. introduced pre-closed sets and investigated their properties. S. N. El-Deebet. al.[3] have defined and studied the notion of pre-closed sets and pre-closure of a set. The concept of generalized pre-closed sets called gp-closed sets was introduced by T. Noiriet. al. [14]. The concept of ideals is treated in the classic text by K. Kuratowski [8] and Vaidyanathaswamy [15]. Ideals had significant impact in research in Topology. The articles of Hamlet and Jankovic [5-6] initiated the applications of topological ideals in the generalization of fundamental properties in General Topology. Newcomb [12] applied topological ideals to generalize the most basic properties in general topology. In this paper, we introduce and investigate a new class of closed sets with respect to an ideal called Igp-closed sets. We investigate their properties and the relationship of these sets. Moreover, by using these sets we define new classes of maps and obtain analogues of closed maps and continuous maps in ideal topological spaces.

Preliminaries

A non-empty collection I of subsets on a topological space (X,τ) is called a topological ideal [5] if it satisfies following two conditions:

(i) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity),

(ii) $A \in I$ and $B \in I$ implies $A \bigcup B \in I$ (finite additivity).

We denote a topological space (X,τ) with an ideal I defined on X by (X,τ,I). If (X,τ,I). is an ideal space, (Y,σ) is a topological space and $f:(X,\tau,I)\to(Y,\sigma)$ is a function, then $f(I)=\{f(I_1):I_1\in I\}$ is an ideal of Y [12]. If I is ideal of subsets of X and Y is subset of X, then $I_Y=\{Y\bigcap I_1:I_1\in I\}$ is an ideal of subsets of Y [12]. If $f:(X,\tau)\to(Y,\sigma,I)$ is an injection then $f^{-1}(I)=\{f^{-1}(B):B\in I\}$ is an ideal on X [12]. Let (X,τ) be a topological space. A subset A of X is said to be pre-open set [10] if $A\subseteq \text{int}(clA)$ and a pre-closed set if $cl(\text{int}\,A)\subseteq A$. Union of pre-open sets is pre-open [11]. Intersection of pre-open set and open set is pre-open [13]. The intersection of all pre-closed sets containing a subset A of a space X is called pre-closure [3] of A and is denoted by pcl(A). Also $pcl(A)=A\bigcup cl(\text{int}\,A)$. The union of all pre-open sets which are contained in A is called pre-interior [11] of A and is denoted by pint (A). Also $p\,\text{int}(A)=A\bigcap \text{int}(clA)$. For a subset A of X, a point x in X is a pre-limit point [11] of A if every pre-open set containing x intersect A in a point different from x. The set of all pre-limit points of A is called the pre-derived set of A and is denoted by $D_P[A]$ [11]. In a topological space (X,τ), if A is subset of X then $D_P[A]\subseteq D[A]$ and $pcl(A)=A\bigcup D_P[A]$.

Let (X,τ) be a topological space. A subset A of X is said to be generalized-closed set (briefly g-closed) [9] if $clA\subseteq U$ whenever $A\subseteq U$ and U open in X. A is generalized-open (briefly, g-open) if its complement (X – A) is generalized-closed. Every closed set is g-closed. A subset A of a space X is called generalized pre-closed (briefly, gp-closed) [14] if $pclA\subseteq U$ whenever $A\subseteq U$ and U is open in X. A is generalized pre-open (briefly, gp-open) if its complement (X – A) is generalized pre-closed. A function $f:(X,\tau)\to(Y,\sigma)$ is called generalized continuous (briefly g-continuous) [2] if $f^{-1}(G)$ is g-closed in X for every closed set G of Y. A function $f:(X,\tau)\to(Y,\sigma)$ is called generalized pre-continuous (briefly gp-continuous) [14] if $f^{-1}(G)$ is gp-closed in X for every closed set G of Y. Let $x\in X$. A subset $U\subseteq X$ is called pre-neighborhood [10] of x in X if there exists pre-open set A such that $x\in A\subset U$. A function $f:(X,\tau)\to(Y,\sigma)$ is called gp-irresolute [1] if $f^{-1}(G)$ is gp-closed in X for every gp-closed set G of Y.

For subsets A and B of X, the following assertions [7] are valid

1. pint (X - A) = X - pcl(A)
2. pcl (X - A) = X - pint(A)
3. pcl(A) $\subset$ cl(A)
4. pint(A $\cap$ B) $\subset$ pint(A) $\cap$ pint(B)
5. pint(A) $\cup$ pint(B) $\subset$ pint(A $\cup$ B)
6. pcl(pcl(A)) = pcl(A)

By a space, we always mean a topological space (X,τ) with no separation axioms assumed. The closure and interior of A are denoted by cl(A) and int(A) respectively.

Gp-Closed Sets Via Ideals

In this section, we introduce and investigate the concept of generalized pre-closed sets with respect to an ideal (briefly Igp-closed sets) which is an extension of generalized pre-closed sets defined by Noiriet. al. [14].

Definition 3.1. Let (X,τ) be a topological space and I be an ideal on X. A subset A of X is said to be generalized pre-closed with respect to an ideal (briefly, Igp-closed) if and only if $pcl(A)-B\in I$, whenever $A\subset B$ and B is open in X. A subset $A\subset X$ is said to be generalized pre-open with respect to an ideal (briefly Igp-open) if and only if (X – A) is Igp-closed.

From above definition 3.1 we have following implication. Closed $\Rightarrow$ pre-closed $\Rightarrow$ gp-closed $\Rightarrow$ Igp-closed.

Theorem 3.1. If A and B are Igp-closed sets of

(X,τ,I) such that $D[A]\subseteq D_P[A]$ and $D[B]\subseteq D_P[B]$, then $A\cup B$ is also Igp-closed.

Proof. Let A and B be Igp-closed subsets of (X,τ,I) such that $D[A]\subseteq D_P[A]$ and $D[B]\subseteq D_P[B]$. As for any subset A, $D_P[A]\subseteq D[A]$. Therefore $D_P[A]=D[A]$ and $D_P[B]=D[B]$. That is cl(A)= pcl(A) and cl(B)= pcl(B). Let $A\cup B\subset U$ and U open, then $A\subset U$ and $B\subset U$. Since A and B are Igp-closed, $pcl(A)-U\in I$ and $pcl(B)-U\in I$.

Now $pcl(A\cup B)-U=cl(A\cup B)-U=(cl(A)\cup cl(B))-U=(pcl(A)\cup pcl(B))-U=(pcl(A)-U)\cup(pcl(B)-U)\in I$.

So $pcl(A\cup B)-U\in I$, thereby implying that $A\cup B$ is Igp-closed.

Corollary 3.1. If A and B are Igp-open sets in (X,τ,I) such that $D[X-A]\subseteq D_P[X-A]$ and $D[X-B]\subseteq D_P[X-B]$,, then $A\cap B$ is Igp-open.

Theorem 3.2. If A is Igp-closed and $A\subset B\subset pcl(A)$ in

X, then B is Igp-closed in (X,τ,I).

Proof. Suppose A is Igp-closed and $A\subset B\subset pcl(A)$ in (X,τ,I). Let $B\subset U$ and U be open in X.

Then $A \subset U$ and U is open in X. Since A is Igp-closed, implies $pcl(A) - U \in I$. Now therefore $pcl(B) \subset pcl(pcl(A)) = pcl(A)$. This implies $pcl(B) - U \subset pcl(A) - U \in I$. Hence $pcl(B) - U \in I$ for $B \subset U$ and U is open in X, implying thereby that B is Igp-closed.

Corollary 3.2. If pint(A) $\subset$ B $\subset$ A and A is Igp-open in X, then B is Igp-open in X.

Theorem 3.2. Let $A \subset Y \subset X$ and suppose that A is Igp-closed in (X, τ, I). Then A is Igp-closed relative to the subspace Y of X with respect to the ideal

$I_Y = \{F \subset Y : F \in I\} = \{Y \bigcap F : F \in I\}$.

Proof. Suppose $A \subset Y \bigcap U$ and U is open in X, then $A \subset U$. Since A is Igp-closed in (X, τ, I) we have

$pcl(A) - U \in I$. Now

$(pcl(A) \bigcap Y) - (U \bigcap Y) = (pcl(A) - U) \bigcap Y \in I \bigcap Y = I_Y$,

whenever $A \subset Y \bigcap U$ and U is open in X. Hence A is Igp-closed relative to the subspace Y.

Theorem 3.4. A set A is Igp-open in (X, τ, I) if and only if (F – U) $\subset$ pint(A) for some U $\in$ I whenever F $\subset$ A and F is closed.

Proof. Suppose A is Igp-open in (X, τ, I). Let F $\subset$ A and F be closed. So, X-A $\subset$ X-F. By definition, pcl(X - A) - (X - F) $\in$ I that is

pcl(X - A) $\subset$ (X - F) $\bigcup$ U for some U $\in$ I.

This implies X - ((X - F) $\bigcup$ U) $\subset$ X - (pcl(X - A)).

Hence (F – U) $\subset$ pint(A).

Conversely, suppose that F $\subset$ A and F is closed implies (F – U) $\subset$ pint(A), for some U $\in$ I.

Let G be an open set such that X - A $\subset$ G. Then X - G $\subset$ A.

Therefore, (X - G) - U $\subset$ pint(A) = X - pcl(X - A).

This gives that X - (G $\bigcup$ U) $\subset$ X - pcl(X - A). Then, pcl(X - A) $\subset$ G $\bigcup$ U, for some U $\in$ I. This shows that pcl(X - A) - G $\in$ I. Hence X - A is Igp-closed.

Theorem 3.5. If A and B are separated Igp-open sets in (X, τ, I), then $A \bigcup B$ is Igp-open.

Proof. Since A and B are separated Igp-open sets in (X, τ, I) and F be a closed subset of $A \bigcup B$. Since A and B are separated sets so cl(A) $\bigcap$ B= ϕ and A $\bigcap$ cl(B)= ϕ.

Now, F $\bigcap$ cl(A) $\subset$ (A $\bigcup$ B) $\bigcap$ cl(A) $\subset$ A $\bigcup$ ϕ =A. Similarly, F $\bigcap$ cl(B) $\subset$ B. By theorem 3.4.,

(F $\bigcap$ cl(A)) - U_1 $\subset$ pint (A), (F $\bigcap$ cl(B)) - U_2 $\subset$ pint(B), for some $U_1, U_2 \in$ I. This means (F $\bigcap$ cl (A)) - pint(A) $\in$ I and (F $\bigcap$ cl(B)) - pint(B) $\in$ I.

Then [(F $\bigcap$ cl (A))- pint(A)] $\bigcup$ [(F $\bigcap$ cl(B))- pint(B)] $\in$ I.

Hence (F $\bigcap$ (cl(A) $\bigcup$ cl(B)))-(pint(A) $\bigcup$ pint(B)) $\in$ I.

But F = F $\bigcap$ (A $\bigcup$ B) $\subset$ F $\bigcap$ cl(A $\bigcup$ B) and we have

F - pint(A $\bigcup$ B) $\subset$ (F $\bigcap$ (cl(A) $\bigcup$ cl(B))) - pint(A $\bigcup$ B) $\subset$ (F $\bigcap$ (cl(A) $\bigcup$ cl(B))) - (pint(A) $\bigcup$ pint(B)) $\in$ I.

Hence, (F – U) $\subset$ pint(A $\bigcup$ B), for some U $\in$ I. This proves $A \bigcup B$ is Igp-open.

Corollary 3.3. Let A and B be Igp-closed sets in

(X,τ,I) and suppose X - A and X - B are separated sets in X. Then $A\bigcap B$ is Igp-closed.

Theorem 3.6. If $A\subset B\subset X$, A is Igp-open relative to B and B is open in X, then A is Igp-open relative to X.

Proof. Suppose F$\subset$A and F is closed. Since A is Igp-open relative to B, by theorem 3.4, (F - U_1)$\subset$pint$_B$(A) for some $U_1\in$I. This implies there exists a set G_1 pre-open in X such that F - $U_1\subset G_1\bigcap$B$\subset$A for some $U_1\in$I. Since B is open, this implies B is Igp-open. As F$\subset$B and F is closed, then F - $U_2\subset$pint (B) for some $U_2\in$I. This implies there exists a set G_2 pre-open in X such that F - $U_2\subset G_2\subset$B for some $U_2\in$I. Now F - ($U_1\bigcup U_2$) = (F - U_1)$\bigcap$ (F - U_2)$\subset$($G_1\bigcap$ B)

$\bigcap G_2$ $=G_1\bigcap$B, because $G_2\subset$B. So F - ($U_1\bigcup U_2$)$\subset G_1\bigcap$B$\subset$A. As $G_1\bigcap$B is pre-open in X, this implies F - ($U_1\bigcup U_2$)$\subset$ pint(A) for some ($U_1\bigcup U_2$)$\in$I. Hence A is Igp-open relative to X.

IGP-Continuous Maps And Igp-Closed Maps

Having discussed generalized pre-closed sets with respect to an ideal, we now turn to introduce the concepts of generalized pre-continuous maps with respect to an ideal (briefly Igp-continuous), generalized pre-closed maps with respect to an ideal (briefly Igp-closed) and Igp-irresolute.

Definition 4.1.A map $f:(X,\tau,I)\to(Y,\sigma)$ is called generalized pre-continuous with respect to an Ideal (briefly Igp-continuous) if $f^{-1}(G)$ is Igp-open in X for every open set G of Y.

Definition 4.2.A map $f:(X,\tau,I)\to(Y,\sigma,J)$ is called generalized pre-closed with respect to an ideal briefly Igp-closed (generalized pre-open with respect to an ideal briefly Igp-open) if the image of every closed set (open set) in X is Jgp-closed (Jgp-open) in Y.

Definition 4.3.A map $f:(X,\tau,I)\to(Y,\sigma,J)$ is called Igp-irresolute if $f^{-1}(G)$ is Igp-open in X for every Jgp-open subset G of Y.

Definition 4.4.A map $f:(X,\tau,I)\to(Y,\sigma,J)$ is called Igp-resolute if the image of every Igp-closed set in X is Jgp-closed in Y.

Theorem 4.1. Let $f:(X,\tau,I)\to(Y,\sigma)$ be any map. Then

$(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)$.

(i) f isIgp-continuous.

(ii) The inverse image of each closed set in (Y,σ) is Igp-closed.

(iii) For each x$\in$X and each G$\in\sigma$ containing f (x), there exists an Igp-open set W containing x such that f (W) $\subseteq$G.

(iv) For each x$\in$X and open set G in Y with

f (x) $\in$G, $f^{-1}(G)$ is an Igp-neighborhood of x.

Proof. $(i)\Rightarrow(ii)$ Let G be closed in Y. Then B = Y - G is open in Y. So, $f^{-1}(B)=f^{-1}(Y)$ - $f^{-1}(G)$ = X - $f^{-1}(G)$. By (i) $f^{-1}(B)$ is Igp-open in X.

Hence $f^{-1}(B)$ is Igp-closed in X.

$(ii)\Rightarrow(iii)$ Let x$\in$X and let G be open set in Y such that

f (x) $\in$ G. Then B = Y - G is closed in Y and f (x)$\notin$ B. By (ii) $f^{-1}(B)=f^{-1}(Y)$ - $f^{-1}(G)$ =X - $f^{-1}(G)$ is Igp-closed in X. So $f^{-1}(G)$ is Igp-open in X. Let W = $f^{-1}(G)$, we have x$\in$W and f (W) $\subseteq$G.

$(iii)\Rightarrow(iv)$ Let G be open set in Y and let f (x) $\in$ G.

Then by (iii), there exists an Igp-open set W containing x

such that f (W) $\subseteq$ G. So x $\in$ W $\subset$ $f^{-1}(G)$ Hence $f^{-1}(G)$ is an Igp-neighborhood of x.

The following theorem 4.2 for Igp-irresolute is an analogue of the theorem 4.1.

Theorem 4.2. Let $f:(X,\tau,I)\to(Y,\sigma,J)$ be any map, then $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)$.

(i) f is Igp-irresolute.

(ii) The inverse image of each Igp-closed set in (Y,σ,J) is Igp-closed in (X,τ,I).

(iii) For each x $\in$ X and each Igp-open set G containing

f (x), there exists an Igp-open set W containing x such that f (W) $\subseteq$ G.

(iv) For each x $\in$ X and Igp-open set G in Y with f (x) $\in$ G, $f^{-1}(G)$ is an Igp-neighborhood of x.

The following theorem 4.3 for Igp-closed maps and Igp-resolute maps is an analogue of theorem 11.2 of Dugundji [4] for closed maps.

Theorem 4.3. If $f:(X,\tau,I)\to(Y,\sigma,J)$ is Igp-closed

(Igp-resolute) map then for each y $\in$ Y and each open set (Igp-open set) U containing $f^{-1}(y)$, there is a Jgp-open set V of Y such that y $\in$ V and $f^{-1}(V)\subset$ U.

Proof. We give the proof of non-parenthesis part. The proof of parenthesis part is similar. Let f be Igp-closed map. If y $\in$ Y and U is any open set in X containing $f^{-1}(y)$, then (X – U) is closed set in X and (X - U) $\cap$ $f^{-1}(y) = \phi$. Since f is Igp-closed, f (X - U) is Jgp-closed in Y and

y $\notin$ f (X - U). Let V = Y - f (X - U), then V is Jgp-open set in Y containing y and $f^{-1}(V)\subset$ U.

Theorem 4.4. If $f:(X,\tau,I)\to(Y,\sigma,J)$ is a bijection, then following are equivalent:

(i) $f^{-1}:(Y,\sigma,J)\to(X,\tau,I)$ is Jgp-continuous.

(ii) f is Igp-open map.

(iii) f is Igp-closed map.

Proof. $(i)\Rightarrow(ii)$ Let U be open set of X. Since f^{-1} is Jgp-continuous, $(f^{-1})^{-1}$(U) = f (U) is Jgp-open in Y. So f is Igp-open.

$(ii)\Rightarrow(iii)$ Let U be closed set of X. Then (X - U) is open set in X. By (ii) f (X - U) is Jgp-open in Y. Therefore f (X - U) = Y - f (U) is Jgp-open in Y. Here f (U) is Jgp-closed in Y. Hence f is Igp-closed.

$(iii)\Rightarrow(i)$ Let U be closed set of X. By (iii) f (U) is Jgp-closed in Y. Since f (U) = $(f^{-1})^{-1}$(U) that implies f^{-1} is Jgp-continuous.

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